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SCHRAMM-LOEWNER EVOLUTIONS (D-MATH)
EXERCISE SHEET 2

Throughout this exercise sheet, whenever γ is a curve, we write ξ for its driving function. Furthermore, let (g_t) be the mapping out functions, (K_t) the chordal hulls and (ζ_z) be the swallowing times.

Exercise 1. The goal of this question will be to prove a characterization result for squared Bessel processes. We suppose that X^x is a continuous process started from x and taking values in $(0, \infty)$ for each $x > 0$. We assume that the family (X^x) defines a Markov process and that it satisfies the scaling property

$$X^x \overset{d}{=} (\lambda X^{x/\lambda}_{t/\lambda} \colon t \geq 0) \quad \text{for all } \lambda, x > 0 \; .$$

Our goal will be to show that (X^x) is (up to rescaling) a squared Bessel process of dimension $\delta \geq 2$.

(i) Show that (X^x) defines a strong Markov process.

(ii) For $x > 0$ let

$$\begin{aligned} \sigma^x_t &= \int_0^t du / X^x_u \quad \text{for } t \geq 0 \; , \\ \tau^x_s &= \inf \{ t \geq 0 \colon \sigma^x_t \geq s \} \quad \text{for } s \geq 0 \; , \\ P^x &= \log(X^x \circ \tau^x) \; . \end{aligned}$$

Show that the process P^x has independent and stationary increments.

(iii) Deduce that there are constants $\mu \in \mathbb{R}$ and $\sigma \geq 0$ such that

$$P^x = (\log(x) + \mu s + \sigma W^x_s \colon s \geq 0) \quad \text{for all } x > 0$$

where W^x is a standard Brownian motion for each $x > 0$.

(iv) Deduce that X^x satisfies the SDE

$$\begin{aligned} dX^x_t &= (\mu + \sigma^2/2)\, dt + \sigma \sqrt{X^x_t}\, dB^x_t \quad \text{where} \\ B^x &:= \int_0^{\cdot} \sqrt{X^x_u}\, d(W^x \circ \sigma^x)_u \quad \text{is a standard Brownian motion} \; . \end{aligned}$$

(v) If $\sigma = 0$ then X^x is just a deterministic affine function and so we suppose that $\sigma > 0$. Let $\delta = 2 + 4\mu/\sigma^2$ and $\lambda = 4/\sigma^2$. Show that $\delta \geq 2$ and that λX^x is a squared Bessel process of dimension δ started from x.

Exercise 2. In this exercise, we will classify certain conformally invariant random curves. Let γ be a random curve starting at 0 generated by a Loewner chain with driving function ξ such that $\gamma([0,\infty)) \cap (-\infty,-1] = \emptyset$. Let $O_t = g_t(-1)$ which we call a marked point. We now assume that for $t \geq 0$ conditionally on $\gamma|_{[0,t]}$, the curve

$$\left(\frac{g_t(\gamma_{t+(\xi_t - O_t)^2 s}) - \xi_t}{\xi_t - O_t} : s \geq 0 \right)$$

has the same law as γ. This is a conformal Markov property with a marked point. Note that the time rescaling factor $(\xi_t - O_t)^2$ appears only to ensure that the curve is parameterized by halfplane capacity. We also suppose that γ is not a deterministic curve.

(i) Let $Y = (\xi - O)^2$. Show that Y has the property that for $t \geq 0$ conditionally on $Y|_{[0,t]}$ the process $(Y_{t+Y_t s}/Y_t : s \geq 0)$ has the same law as Y.

(ii) Use exercise 1 to show that there exists $\delta \geq 2$ and $\kappa > 0$ such that $Y = \kappa X$ where X is a squared Bessel process starting from $1/\kappa$ and satisfying the SDE

$$dX_t = \delta\, dt + 2\sqrt{X_t}\, dB_t$$

where B is a standard Brownian motion.

(iii) Deduce that (ξ, O) satisfy the following system of SDEs

$$d\xi_t = \frac{\rho}{\xi_t - O_t}\, dt + \sqrt{\kappa}\, dB_t\ ,$$
$$dO_t = \frac{2\, dt}{O_t - \xi_t}$$

where $\rho = (\delta - 1)\kappa/2 - 2\, (\geq \kappa/2 - 2)$.

It turns out that the solution to the SDE in part (iii) indeed generates a continuous curve the law of which we call $\mathrm{SLE}_\kappa(\rho)$; in fact $\mathrm{SLE}_\kappa(\rho)$ can be defined whenever $\rho > -2$.

Exercise 3. Fix $\alpha > 0$. Whenever K is a compact chordal hull satisfying $0 \notin K$ we write $\Phi_K : \mathbb{H} \setminus K \to \mathbb{H}$ for the unique conformal transformation with $\Phi_K(0) = 0$ and $\Phi_K(z)/z \to 1$ as $|z| \to \infty$. Let E be a curve from 0 to ∞ in $\overline{\mathbb{H}}$ satisfying

$$\mathbb{P}(E([0,\infty)) \cap K = \emptyset) = \Phi_K'(0)^\alpha \quad \text{for all chordal hulls with } 0 \notin K\ .$$

Also define $E' = (|\Re(E)| + i\Im(E))^2$ and let E'' be an independent copy of E'.

(i) Fix a compact chordal hull A such that $A \cap (-\infty, 0] = \emptyset$. Show that

$$\mathbb{P}(E'([0,\infty)) \cap A = \emptyset) = \mathbb{P}(E([0,\infty)) \cap (\sqrt{A} \cup A') = \emptyset) = \Phi'_{\sqrt{A}\cup A'}(0)^\alpha$$

where A' denotes the reflection of $\sqrt{A}$ across the imaginary axis.

(ii) Show that $\Phi_{\sqrt{A}\cup A'}(\epsilon) = \Phi_A(\epsilon^2)^{1/2}$ for $\epsilon > 0$ sufficiently small and deduce that

$$\mathbb{P}(E'([0,\infty)) \cap A = \emptyset) = \Phi_A'(0)^{\alpha/2}\ .$$

Hint: Use Schwarz reflection to write down $\Phi_{\sqrt{A}\cup A'}$ in terms of Φ_A.

(iii) Show that the right boundary of the set $E'([0,\infty)) \cup E''([0,\infty))$ has the same law as the right boundary of $E([0,\infty))$.

In the special case where E is a Brownian excursion from 0 to ∞ in $\mathbb{H}$ we have $\alpha = 1$ and the process E' is called a Brownian excursion from 0 to ∞ in $\mathbb{H}$ with perpendicular reflection along $(-\infty, 0)$.

Submission of solutions. Send your solutions via email to Matthis Lehmkuehler by Monday 20/12/21 at 5 p.m.