Wahrscheinlichkeitstheorie und Statistik

Lösungen Serie 2

Version 1 (25. Februar 2022)

Bitte stellt Fragen in den Übungen und/oder im Forum des Moodle-Kurs (und/oder (anonym) in diesem file https://docs.google.com/document/d/1CfTkwrNOhTKB8y8cVQW_deUahCejTbQWlm_BzrHIFYA/edit?usp=sharing)

Wir empfehlen die Aufgaben selbständig zu lösen und dann im Fach der entsprechenden Übungsgruppe im Raum HG G 53 abzugeben oder selbst mit dieser Lösung zu vergleichen am besten rechtzeitig vor der Übung am **07. März**.

Aufgabe 2.1 One coin is flipped and one die is rolled.

(a) Define a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ space using a Laplace model.

(b) Define random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ on this probability space such that X and Y represent the outcome of flipping the coin and of the roll of the die, respectively.

(c) Show that $\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x]\,\mathbb{P}[Y = y]$ for all $x, y \in \mathbb{R}$. (This means that the random variables X and Y are independent; see later.)

Lösung 2.1

(a) $\Omega := \{0,1\} \times \{1,2,3,4,5,6\}$, $\mathcal{F} := \mathcal{P}(\Omega)$, $\mathbb{P} : \mathcal{F} \to [0,1], A \mapsto \mathbb{P}[A] = \frac{|A|}{|\Omega|} = \frac{|A|}{12}$.

(b) $X : \Omega \to \mathbb{R}, \omega \mapsto X(\omega) := \omega_1$ and $Y : \Omega \to \mathbb{R}, \omega \mapsto Y(\omega) := \omega_2$.

(c) First we compute the left-hand side as

$$\mathbb{P}[X = x, Y = y] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}] = \frac{|\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}|}{12},$$

where the numerator can be simplified as

$$|\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}| = \begin{cases} 1, & (x,y) \in \Omega, \\ 0, & (x,y) \notin \Omega. \end{cases}$$

This results in

$$\mathbb{P}[X = x, Y = y] = \begin{cases} \frac{1}{12}, & (x,y) \in \Omega, \\ 0, & (x,y) \notin \Omega. \end{cases}$$

Now, we consider the right-hand side by starting with calculating the probability

$$\mathbb{P}[X = x] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x\}] = \frac{|\{\omega \in \Omega : X(\omega) = x\}|}{12},$$

where the set appearing in the numerator can be expressed as

$$\{\omega \in \Omega : X(\omega) = x\} = \begin{cases} \{(x,1),(x,2),(x,3),(x,4),(x,5),(x,6)\}, & x \in \{0,1\}, \\ \varnothing, & x \notin \{0,1\}. \end{cases}$$

So by counting, we obtain

$$\mathbb{P}[X = x] = \begin{cases} \frac{6}{12} = \frac{1}{2}, & x \in \{0,1\}, \\ 0, & x \notin \{0,1\}. \end{cases}$$

Analogously, we get

$$\mathbb{P}[Y = y] = \begin{cases} \frac{2}{12} = \frac{1}{6}, & y \in \{1,2,3,4,5,6\}, \\ 0, & y \notin \{1,2,3,4,5,6\}. \end{cases}$$

Since $(x,y) \in \Omega$ is equivalent to $(x \in \{0,1\} \ \wedge \ y \in \{1,2,3,4,5,6\})$, we can conclude that

$$\mathbb{P}[X = x]\,\mathbb{P}[Y = y] = \begin{cases} \frac{1}{2}\frac{1}{6} = \frac{1}{12}, & (x,y) \in \Omega, \\ 0, & (x,y) \notin \Omega. \end{cases}$$

Aufgabe 2.2 We have two dice. One is ordinary with the numbers $1, 2, 3, 4, 5, 6$ and one is special where 6 is replaced by 7 (i.e. $1, 2, 3, 4, 5, 7$). We flip a coin to decide which die is rolled. If flipping the coin results in heads, the ordinary die is rolled, otherwise the special die is rolled.

(a) Define a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ using a Laplace model.

(i) Define random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ such that X and Y represent the outcomes of flipping the coin and of the roll of the die, respectively.

(ii) What is the cardinality $|\mathcal{F}|$? Give examples of events $E_1, E_2, E_3, E_4 \in \mathcal{F}$ such that $\mathbb{P}[E_i] \neq \mathbb{P}[E_j], \ \forall i \neq j$.

(b) What is the probability that rolling the die results in an even number?

(c) Show that $\mathbb{P}[X = x, Y = y] \neq \mathbb{P}[X = x]\,\mathbb{P}[Y = y]$ for some $x, y \in \mathbb{R}$. (This means that the random variables X and Y are not independent; see later and cf. Aufgabe 2.1.)

Lösung 2.2

(a) $\Omega := \{0, 1\} \times \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} := 2^\Omega$, $\mathbb{P} : \mathcal{F} \to [0, 1], A \mapsto \mathbb{P}[A] = \frac{|A|}{|\Omega|} = \frac{|A|}{12}$. Alternatively one could define $\Omega := \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 7)\}$. Then Y can be defined more simply as $Y(\omega) := \omega_2$.

(i) $X : \Omega \to \mathbb{R}, \omega \mapsto X(\omega) := \omega_1$ and $Y : \Omega \to \mathbb{R}, \omega \mapsto Y(\omega) := \begin{cases} \omega_2, & \omega_2 \leq 5, \\ 6, & \omega_2 = 6 \text{ and } \omega_1 = 0, \\ 7, & \omega_2 = 6 \text{ and } \omega_1 = 1. \end{cases}$ An elegant way to write this is $Y(\omega) := \omega_2 I_{\{\omega_2 \leq 5\}} + (\omega_1 + \omega_2) I_{\{\omega_2 > 5\}}$.

(ii) $|\mathcal{F}| = 2^{12} = 4096$, so there are many different possibilities to choose the examples—e.g. $E_1 = \varnothing, E_2 = \{(0, 1)\}, E_3 = \{(0, 1), (0, 3), (1, 2)\}, E_4 = \Omega$ with $\mathbb{P}[E_1] = 0$, $\mathbb{P}[E_2] = \frac{1}{12}$, $\mathbb{P}[E_3] = \frac{3}{12}$, $\mathbb{P}[E_4] = 1$.

(b) We can directly count that $|\{\omega \in \Omega : \exists k \in \mathbb{N} \text{ with } Y = 2k\}| = 3 + 2 = 5$. Hence, the probability is $\mathbb{P}[\{\omega \in \Omega : \exists k \in \mathbb{N} \text{ with } Y = 2k\}] = \frac{5}{12}$.

(c) To show the dependence of the two random variables, it is sufficient to find one counterexample for the equation. A simple choice is $x = 0$ and $y = 7$. First we compute the left-hand side as

$$\mathbb{P}[X = 0, Y = 7] = \frac{|\{\omega \in \Omega : X(\omega) = 0 \text{ and } Y(\omega) = 7\}|}{12} = 0.$$

Now, we consider the right-hand side by starting with calculating the probability

$$\mathbb{P}[X = 0] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = 0\}] = \frac{|\{\omega \in \Omega : X(\omega) = 0\}|}{12} = \frac{6}{12} = \frac{1}{2}.$$

Analogously, we get

$$\mathbb{P}[Y = 7] = \mathbb{P}[\{\omega \in \Omega : Y(\omega) = 7\}] = \frac{|\{\omega \in \Omega : Y(\omega) = 7\}|}{12} = \frac{1}{12}.$$

So we can conclude that

$$\mathbb{P}[X = 0, Y = 7] = 0 \neq \frac{1}{24} = \mathbb{P}[X = 0]\,\mathbb{P}[Y = 7].$$

Aufgabe 2.3

(a) Seien A und B zwei Ereignisse mit

$$\mathbb{P}[A^c] = \frac{1}{2}, \qquad \mathbb{P}[B^c] = \frac{1}{2}, \qquad \mathbb{P}[A^c \cap B^c] = p.$$

Bestimmen Sie als Funktion von p

(i) die Wahrscheinlichkeiten $\mathbb{P}[A \cap B]$, $\mathbb{P}[A \cap B^c]$ und $\mathbb{P}[A^c \cap B]$. In welchem Bereich darf p liegen?

(ii) die Wahrscheinlichkeit, dass höchstens i der beiden Ereignisse A und B eintreten, wobei $i = 0, 1, 2$.

Lösung 2.3

(a) (i) Da

$$(A \cup B)^c = (A^c \cap B^c),$$

bekommt man mit Hilfe der Formel des Skripts

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \\ &= [1 - \mathbb{P}(A^c)] + [1 - \mathbb{P}(B^c)] - [1 - \mathbb{P}(A^c \cap B^c)] \\ &= \frac{1}{2} + \frac{1}{2} - (1 - p) \\ &= p.\end{aligned}$$

Da $A \cap B^c$ und $A \cap B$ disjunkt sind mit $(A \cap B^c) \cup (A \cap B) = A$, erhalten wir

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = \frac{1}{2} - p.$$

Ähnlich bekommt man

$$\mathbb{P}(A^c \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{1}{2} - p.$$

Da alle Wahrscheinlichkeiten in $[0, 1]$ liegen müssen, muss p im Bereich $[0, \frac{1}{2}]$ liegen.

(ii) Wir definieren das Ereignis

$$C_i := \text{„Höchstens } i \text{ der beiden Ereignisse } A \text{ und } B \text{ treten ein.“},$$

für alle $i = 0, 1, 2$. Wir erhalten

$$\mathbb{P}(C_0) = \mathbb{P}(A^c \cap B^c) = p,$$

$$\begin{aligned}\mathbb{P}(C_1) &= \mathbb{P}(C_0) + \mathbb{P}(A \cup B) - \mathbb{P}(A \cap B) \\ &= \mathbb{P}(C_0) + 1 - \mathbb{P}(A^c \cap B^c) - \mathbb{P}(A \cap B) \\ &= p + 1 - p - p \\ &= 1 - p\end{aligned}$$

und

$$\mathbb{P}(C_2) = \mathbb{P}(C_1) + \mathbb{P}(A \cap B) = 1 - p + p = 1.$$

Wenn du Feedback zum Übungszettel hast, schreibe bitte eine Mail an Jakob Heiss.