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Algebraic Topology II
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Problem set 1

1. Consider the space $X := \Delta^n/\partial\Delta^n \approx S^n$ for $n \geq 0$. Denote by $* \in X$ the point corresponding to $\partial\Delta^n$. The quotient map $\sigma_n : \Delta^n \to X$, viewed as a singular n-simplex, is a cycle in $S_n(X, *)$.

   (a) Show that $[\sigma_n]$ generates $H_n(X, *; \mathbb{Z}) \cong \widetilde{H}_n(X; \mathbb{Z}) \cong \mathbb{Z}$.

   (b) Let G be an abelian group. Then for all $g \in G$, $g \cdot \sigma_n$ is a cycle in $S_n(X, *; G)$. Show that the map

   $$\begin{aligned} G &\longrightarrow H_n(X, *; G) \\ g &\longmapsto [g \cdot \sigma_n] \end{aligned}$$

   is an isomorphism.

2. Consider the space $Y \approx S^n$ obtained by gluing two copies $\Delta^n_\pm$ of Δ^n along their boundaries (using the identity map). Consider the obvious singular simplices $\tau_\pm : \Delta^n \to Y$ mapping to the subsets $\Delta^n_\pm \subset Y$. Check that $\tau_+ - \tau_-$ is a cycle and prove that $[\tau_+ - \tau_-]$ generates $\widetilde{H}_n(Y; \mathbb{Z})$.

   Hint: Use the Mayer-Vietoris sequence.

3. Let $\pi\colon X \to Y$ be a 2:1 covering. Recall the short exact sequence of chain complexes

   $$0 \to S.(Y; \mathbb{Z}_2) \xrightarrow{T} S.(X; \mathbb{Z}_2) \xrightarrow{\pi_c} S.(Y; \mathbb{Z}_2) \to 0$$

   and its associated long exact sequence in homology. (These sequences are called *Smith* short/long exact sequence.) Show that

   (a) $T \circ \pi_c = id + \Theta_c$, where $\Theta\colon X \to X$ is the unique non-trivial deck transformation of π.

   (b) Assume that $H_i(X; \mathbb{Z}_2) = \mathbb{Z}_2$ for some i. Show that $T_* \circ \pi_* = 0$ in degree i.

4. Suppose you know that $H_k(\mathbb{R}P^n; \mathbb{Z}_2) = 0$ for all $k > n$. Use the Smith-sequence for this covering to compute $H_k(\mathbb{R}P^n; \mathbb{Z}_2)$ for $0 \leq k \leq n$.

5. Let $f : \mathbb{R}P^n \to \mathbb{R}P^m$ be any map, where $n > m > 0$. Show that the induced map $f_\# : \pi_1(\mathbb{R}P^n) \to \pi_1(\mathbb{R}P^m)$ is trivial.

6. Show that $\mathbb{R}P^2$ is not a retract of $\mathbb{R}P^3$.

7. The Borsuk-Ulam theorem says that for every map $f : S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ such that $f(x) = f(-x)$. Give a proof of the theorem based on the following steps:

   (a) Let $g : S^n \to S^n$ be an odd map, i.e., such that $g(-x) = -g(x)$ for all $x \in S^n$. Show that g induces a natural homomorphism from the Smith sequence for $S^n \to \mathbb{R}P^n$ to itself in which all squares commute.

   (b) Conclude that every odd $g : S^n \to S^n$ has odd degree.

   (c) Conclude the proof of the theorem.

8. Use Borsuk-Ulam to prove that whenever there exists a map $\phi : S^n \to S^m$ which is equivariant with respect to the antipodal maps, then $n \leq m$.

9. Use Borsuk-Ulam to prove the following: Given Lebesgue measurable subsets bounded subsets $A_1, \ldots, A_m$ of $\mathbb{R}^m$, there exists a hyperplane $H \subset \mathbb{R}^m$ which divides each A_i into pieces of equal measure. (This is known as the "Ham Sandwich Theorem".)

10. Consider $\mathbb{R}P^k = S^k/(x \sim -x)$, and denote by $q\colon S^k \to \mathbb{R}P^k$ the quotient map. View $\mathbb{R}P^{k-1}$ as a subspace of $\mathbb{R}P^k$ as follows: Let $S^{k-1}_{Eq} \subset S^k$ be the equator

$$S^{k-1}_{Eq} = \left\{(x_1, \ldots, x_{k+1} \in S^k | x_{k+1} = 0\right\}.$$

Then $q(S^{k-1}_{Eq}) \subset \mathbb{R}P^k$ is homeomorphic in an obvious way to $\mathbb{R}P^{k-1}$- Consider the space $\mathbb{R}P^k/\mathbb{R}P^{k-1}$ and the quotient map $q'\colon \mathbb{R}P^k \to \mathbb{R}P^k/\mathbb{R}P^{k-1}$. Denote by

$$B^k_+ := \left\{(x_1, \ldots, x_{k+1}) \in S^k | x_{k+1} \geq 0\right\} \subset S^k$$

the closed upper hemisphere and similarly by B^k_- the closed lower hemisphere.

   (a) Show that there exists a homeomorphism $\phi\colon \mathbb{R}P^k/\mathbb{R}P^{k-1} \to S^k$ such that the composition of maps

$$f := \left(S^k \xrightarrow{q} \mathbb{R}P^k \xrightarrow{q'} \mathbb{R}P^k/\mathbb{R}P^{k-1} \xrightarrow{\phi} S^k\right)$$

   sends each open hemisphere $Int(B^k_\pm) \subset S^k$ homeomorphically onto $S^k \backslash \{\text{point}\}$.

   (b) Show that $\deg(f) = \pm\left(1 + (-1)^{k+1}\right)$. (The $\pm$ depends on the choice of ϕ.)
   Hint: Use local degrees.

(c) Consider the space

$$\mathbb{R}P^k \cup_{h_\partial} B^{k+1},$$

where the attaching map $h_\partial \colon \partial B^{k+1} = S^k \to \mathbb{R}P^k$ is the quotient map q. Show that there exists a homeomorphism

$$\left(\mathbb{R}P^k \cup_{h_\partial} B^{k+1}, \mathbb{R}P^k\right) \approx \left(\mathbb{R}P^{k+1}, \mathbb{R}P^k\right)$$

which is the identity on $\mathbb{R}P^k$.

(d) Endow $\mathbb{R}P^n$ with the structure of an n-dimensional CW-complex X with one j-cell in each dimension $0 \leq j \leq n$, as follows:

$$\begin{aligned}
&X^{(0)} = \mathbb{R}P^0 = 1 \text{ point},\\
&\dots\\
&X^{(k)} \approx \mathbb{R}P^k,\\
&X^{(k+1)} \approx \mathbb{R}P^k \cup_{h_\partial} B^{k+1} \approx \mathbb{R}P^{k+1},\\
&\dots\\
&X^{(n)} \approx \mathbb{R}P^{n-1} \cup_{h_\partial} B^n \approx \mathbb{R}P^n.
\end{aligned}$$

(e) Consider the cellular chain complex $C_\bullet^{\mathrm{CW}}(X)$ of the CW-complex described in (c). Denote by $e^{(k)}$ the generator of $C_k^{\mathrm{CW}}(X)$, corresponding to the k-dimensional cell, so that $C_k^{\mathrm{CW}}(X) = \mathbb{Z}e^{(k)}$. Calculate the differential $d \colon C_{k+1}^{\mathrm{CW}}(X) \to C_k^{\mathrm{CW}}(X)$.