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Problem set 2

1. Let H, H', H'' and G be Abelian groups and $f : H \to H'$, $g : H' \to H''$ group homomorphisms. Show that f induces a well defined homomorphism $f_{\mathrm{Tor}} : \mathrm{Tor}(H, G) \to \mathrm{Tor}(H', G)$. Moreover show that $\mathrm{id}_{\mathrm{Tor}} = \mathrm{id}$, $(g \circ f)_{\mathrm{Tor}} = g_{\mathrm{Tor}} \circ f_{\mathrm{Tor}}$ and if f is an isomorphism then $(f^{-1})_{\mathrm{Tor}} = (f_{\mathrm{Tor}})^{-1}$.

2. Prove that the sequence in the universal coefficient theorem for homology is natural with respect to chain maps. That is, given a chain map $f : C_* \to D_*$ show that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(C) \otimes G & \longrightarrow & H_n(C;G) & \longrightarrow & \mathrm{Tor}(H_{n-1}(C), G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f_* & & \downarrow & & \\ 0 & \longrightarrow & H_n(D) \otimes G & \longrightarrow & H_n(D;G) & \longrightarrow & \mathrm{Tor}(H_{n-1}(D), G) & \longrightarrow & 0 \end{array}$$

commutes.

Remark: The statement also holds for the universal coefficient theorem for cohomology.

3. Let C_*, D_* be chain complexes of free Abelian groups and assume that $f : C_* \to D_*$ is a quasi-isomorphism, i.e. a chain map such that $f_* : H_*(C) \to H_*(D)$ is an isomorphism. Let G be an Abelian group. Prove the following statements using naturality of the sequences in the universal coefficient theorems.

   (a) $f \otimes \mathrm{id} : C_* \otimes G \to D_* \otimes G$ is a quasi-isomorphism.

   (b) $f^* : \mathrm{Hom}(D_*, G) \to \mathrm{Hom}(C_*, G)$ is a quasi-isomorphism.

4. Show that the splitting $H^n(X; G) \cong \mathrm{Ext}(H_{n-1}(X); G) \oplus \mathrm{Hom}(H_n(X); G)$ whose existence is asserted by the universal coefficient theorem for cohomology *cannot* be natural in X.

   Hint: Consider the map $\phi : \mathbb{R}P^2 \to S^2$ given by collapsing $\mathbb{R}P^1 \subset \mathbb{R}P^2$ to a point.

5. The Klein bottle K has $H_0(K; \mathbb{Z}) \cong \mathbb{Z}$, $H_1(K; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and all other homology groups vanish. Use this to compute the cohomology of K with coefficients in $\mathbb{Z}$ and the cohomology and homology with coefficients in $\mathbb{Z}_p$ for p prime.

6. Let X be a topological space and let $A, B \subset X$ be subsets. Denote by $S_k(A + B) \subset S_k(X)$ the subspace of chains which are sums of simplices entirely contained in A or B. Show that the quotient $S_k(X)/S_k(A + B)$ is free.

7. Let X be a topological space. Show that

$$H_n(X; \mathbb{Q}) \cong H_n(X; \mathbb{Z}) \otimes \mathbb{Q}.$$

   and

$$H^n(X; \mathbb{Z}) \cong \mathrm{Hom}(H_n(X; \mathbb{Z}), \mathbb{Q}).$$

   Hint: Show that $\mathrm{Tor}(A, \mathbb{Q}) = 0$ and $\mathrm{Ext}(A, \mathbb{Q}) = 0$ for any abelian group A and use the universal coefficients theorems.

8. Let A be an abelian group and R a commutative ring. View $A \otimes_{\mathbb{Z}} R$ as an R-module in the obvious way. Consider also $\hom_{\mathbb{Z}}(A, R)$ and $\hom_R(A \otimes_{\mathbb{Z}} R, R)$ and view them as R-modules in the obvious way.

(a) Show that there exists an isomorphism

$$\varphi\colon \hom_{\mathbb{Z}}(A, R) \xrightarrow{\cong} \hom_R(A \otimes_{\mathbb{Z}} R, R)$$

of R-modules, which is natural wrt homomorphisms of abelian groups $A \to A'$.

(b) Let $C_\bullet$ be a chain complex of abelian groups and consider the cochain complexes $\hom_{\mathbb{Z}}(C_\bullet, R)$ and $\hom_R(C_\bullet \otimes_{\mathbb{Z}} R, R)$. Show that the (co-)boundary operators of these cochain complexes are R-linear (so these are cochain complexes of R-modules). Show that the isomorphism φ from (a) can be chosen in this case to be a cochain isomorphism

$$\varphi\colon \hom_{\mathbb{Z}}(C_\bullet, R) \xrightarrow{\cong} \hom_R(C_\bullet \otimes_{\mathbb{Z}} R, R)$$