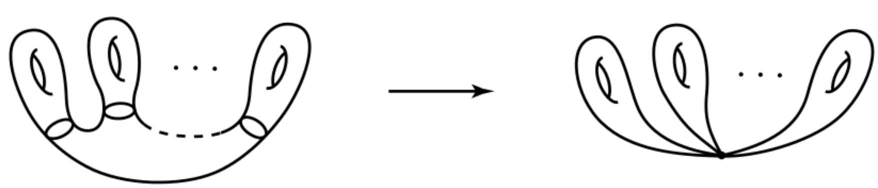
Problem set 3

1. Prove that there are natural ring isomorphisms
$$H^*(\textstyle\coprod_\alpha X_\alpha; R) \to \prod_\alpha H^*(X_\alpha; R), \quad H^*(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha; R) \to \prod_\alpha H^*(X_\alpha, A_\alpha; R),$$
$$\widetilde{H}^*(\textstyle\bigvee_\alpha X_\alpha; R) \cong \prod_\alpha \widetilde{H}^*(X_\alpha; R)$$
for every coefficient group R, where the product on the right is coordinatewise cup product. (Assume in the last case that the spaces are joined at points which are deformation retracts of neighbourhoods.)

2. Show that if $H_n(X;\mathbb{Z})$ is free of finite rank for each n, then $H^*(X;R)$ and $H^*(X;\mathbb{Z}) \otimes R$ are isomorphic as rings.

3. Suppose that a space X can be covered by two acyclic open sets A and B. Using the cup product $H^k(X, A; R) \times H^\ell(X, B; R) \to H^{k+\ell}(X, A \cup B; R)$, show that all cup products of classes in $H^*(X;R)$ of positive dimensions vanish. Generalize to the situation that X can be covered by n acyclic open sets.

4. Compute the cup product structure on $H^*(\Sigma_g;\mathbb{Z})$ for the closed orientable surface Σ_g of genus g, assuming as known the cup product structure on $H^*(T^2;\mathbb{Z})$ and using the map $\pi : \Sigma_g \to \bigvee_g T^2$ depicted below.



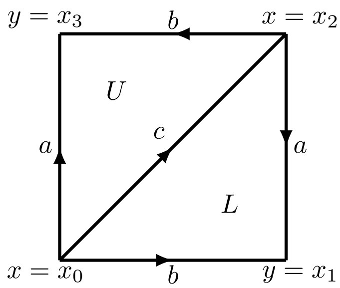
5. Let X, Y be spaces and let $A \subset X$ be a subspace. Denote by $\delta : H^*(A;R) \to H^{*+1}(X, A; R)$ and $\delta' : H^*(A \times Y; R) \to H^{*+1}(X \times Y, A \times Y; R)$ the boundary maps from the LES for the pairs (X, A) and $(X \times Y, A \times Y)$. Prove that $\delta'(a \times b) = \delta(a) \times b$ for $a \in H^k(A;R)$ and $b \in H^\ell(Y;R)$, i.e. that the following diagram commutes:
$$\begin{array}{ccc} H^k(A;R) \times H^\ell(Y;R) & \xrightarrow{\delta \times \mathrm{id}} & H^{k+1}(X, A; R) \times H^\ell(Y;R) \\ \downarrow \times & & \downarrow \times \\ H^{k+\ell}(A \times Y; R) & \xrightarrow{\delta'} & H^{k+\ell+1}(X \times Y, A \times Y; R) \end{array}$$

6. Let $\mu_0 \in H^1(I, \partial I; R) \cong R$ a generator. Prove that the map $H^n(Y;R) \to H^{n+1}(I \times Y, \partial I \times Y; R)$, $\beta \mapsto \beta \times \mu_0$, is an isomorphism for every space Y and every $n \geq 0$. *Hint:* Consider the LES for the pair $(I \times Y, \partial I \times Y)$ and use the result of the previous problem!

7. Show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

8. Let X be the space obtained from $\mathbb{C}P^2$ by attaching a 3-cell by a map $S^2 \to \mathbb{C}P^1 \subset \mathbb{C}P^2$ of degree p, and let $Y = M(\mathbb{Z}_p, 2) \vee S^4$, where $M(\mathbb{Z}_p, 2)$ is S^2 with a 3-cell attached by a map $S^2 \to S^2$ of degree p (this is an example of a *Moore space*). Show that X and Y have isomorphic cohomology rings with $\mathbb{Z}$-coefficients, but not with $\mathbb{Z}_p$-coefficients.

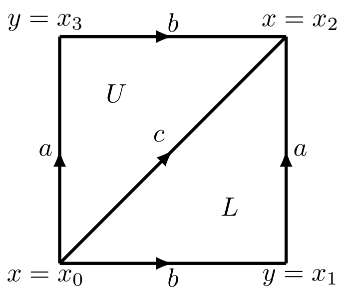
9. Calculate the cup product on $H^*(\mathbb{R}P^2; \mathbb{Z}_2)$ using the following CW-complex structure:



0-cells: $x = x_0 = x_2$, $y = x_1 = x_3$
1-cells: a, b, c
2-cells: $U = [x_0, x_2, x_3]$, $L = [x_0, x_2, x_1]$

Hint: Use the approach described in class on 5.5.2021.

10. Consider the space $X = \mathbb{T}^2$, viewed as a CW-complex as follows:



0-cells: $x = x_0 = x_1 = x_2 = x_3$
1-cells: a, b, c
2-cells: $U = [x_0, x_3, x_2]$, $L = [x_0, x_1, x_2]$

Let $i\colon C_\cdot^{\mathrm{CW}}(X) \to S_\cdot(X)$ be the inclusion. We have seen in class (see the lecture notes from 5.5.2021 on page 4) that, in this specific case, i is a chain map. Show that

$$i_*\colon H_*^{\mathrm{CW}}(X) \longrightarrow H_*(X)$$

is an isomorphism using the following steps:

(a) Show that i_* is an isomorphism on H_0.

(b) Use the Hurewicz theorem (on H_1 and π_1) to show i_* is an isomorphism on H_1.

(c) To show that i_* is an isomorphism on H_2 argue as follows:

i. Put $F := L - U \in C_2^{\mathrm{CW}}(X)$. Show that $[F]$ is a generator of $H_2^{\mathrm{CW}}(X)$, i.e.

$$H_2^{\mathrm{CW}}(X) = \mathbb{Z} \cdot [F].$$

ii. View the cell $U \subset X$ as a subspace, and consider the map $H_2(X) \xrightarrow{j_*} H_2(X, U)$ induced by the inclusion. Put $F' := i(F) \in S_2(X)$. Show that $H_2(X, U) \cong \mathbb{Z}$ and moreover $j_*(F')$ is a generator of $H_2(X, U)$, i.e.

$$H_2(X, U) \cong \mathbb{Z} \cdot j_*(F').$$

Hint for (ii):

$$j_*(F') = j_*[i(L) - i(U)] = [L'] \in H_2(X, U),$$

where $L' \in S_2(X,U)$ is the 2-simplex L, viewed as a relative simplex in $S_2(X,U)$. Now use excision

$$H_2(X,U) \cong H_2(X \backslash \operatorname{Int} U, U \backslash \operatorname{Int} U) = H_2(L, a \cup b \cup c).$$

Note that $(L, a \cup b \cup c)$ is a good pair.

iii. Use (i) and (ii) above to show that $H_2(X) \cong \mathbb{Z}[F']$ and deduce that $i_* \colon H_2^{\mathrm{CW}}(X) \to H_2(X)$ is an isomorphism.

11. Let X be a space, $A \subset X$ a subspace and R a commutative ring with unit.

   (a) Show that the cup product on X induces the following versions of the $\cup$-product:

$$H^p(X;R) \otimes_R H^q(X,A;R) \xrightarrow{\cup} H^{p+q}(X,A;R)$$
$$H^p(X,A;R) \otimes_R H^q(X;R) \xrightarrow{\cup} H^{p+q}(X,A;R)$$
$$H^p(X,A;R) \otimes_R H^q(X,A;R) \xrightarrow{\cup} H^{p+q}(X,A;R)$$

   (b) Let $\delta^* \colon H^k(A;R) \to H^{k+1}(X,A;R)$ be the connecting homomorphism, and consider the restriction maps $i^* \colon H^*(X;R) \to H^*(A;R)$, $j^* \colon H^*(X,A;R) \to H^*(X;R)$ (induced by the inclusions $i \colon A \to X$ and $j \colon (X,\emptyset) \to (X,A)$).
   Show that for all $\alpha \in H^*(A;R), \beta \in H^*(X,A;R)$

$$\delta^*(\alpha \cup i^*\beta) = \delta^*(\alpha) \cup \beta,$$
$$\delta^*(i^*\beta \cup \alpha) = \beta \cup \delta^*(\alpha).$$

   Show that for all $\alpha, \beta \in H^*(X,A;R)$

$$j^*(\alpha \cup \beta) = j^*(\alpha) \cup j^*(\beta).$$

   Show that for all $\alpha \in H^*(X;R), \beta \in H^*(X,A;R)$

$$j^*(\alpha \cup \beta) = \alpha \cup j^*(\beta).$$

12. Use cross products to calculate the cohomology **ring** of $\mathbb{T}^n = S^1 \times \cdots \times S^1$.