Prof. Paul Biran
ETH Zürich

Algebraic Topology II

FS 2022

Problem set 4

1. Show that every covering space of an orientable manifold is an orientable manifold.

2. Show that for a connected non-orientable manifold M there is a unique orientable double cover of M.

3. Show that for any connected closed orientable n-manifold M there is a degree 1 map $f : M \to S^n$.

4. Let $f : M \to N$ be a map between connected closed orientable manifolds and suppose there is a ball $B \subset N$ such that $f^{-1}(B)$ is a disjoint union of open balls $B_1, \ldots, B_k \subset M$ which each get mapped homeomorphically onto B. Show that the degree of f is $\sum \varepsilon_i$, where ε_i is ± 1 according to whether $f|_{B_i} : B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

5. Let M, N be closed connected orientable manifolds and let $f : M \to N$ a p-sheeted covering map. Show that f has degree $\pm p$.

6. Consider a pair of spaces $(X, Y) = (Q \cup R, S \cup T)$ such that $S \subset Q$, $T \subset R$ and such that the interiors of Q, R cover X and the interiors of S, T cover Y. Show that there is a relative Mayer-Vietoris LES

$$\cdots \to H_n(Q \cap R, S \cap T) \to H_n(Q, S) \oplus H_n(R, T) \to H_n(X, Y) \to H_{n-1}(Q \cap R, S \cap T) \to \cdots$$

Hint: Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_n(S \cap T) & \to & S_n(S) \oplus S_n(T) & \to & S_n(S+T) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_n(Q \cap R) & \to & S_n(Q) \oplus S_n(R) & \to & S_n(Q+R) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_n(Q \cap R, S \cap T) & \to & S_n(Q, S) \oplus S_n(R, T) & \to & S_n(Q+R, S+T) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which the horizontal maps are of the form $x \mapsto (x, -x)$ resp. $(x, y) \mapsto x + y$; $S_n(Q+R)$ is the subgroup of $S_n(X)$ consisting of sums of chains in Q and R (and similarly for $S_n(S+T)$), and $S_n(Q+R, S+T)$ denotes the quotient of $S_n(Q+R)$ by $S_n(S+T)$. Show first that the third row is a chain complex. Then show it is exact by considering the diagram as a short exact sequence of chain complexes. Finally deduce the existence of the LES.

7. The goal of this exercise is to prove the following theorem:

   Theorem. *Let M be a connected non-compact manifold of dimension n. Then $H_i(M;R) = 0$ for all $i \geq n$.*

   Let $i \geq n$ and $a = [z] \in H_i(M;R)$. We will omit R from the notation. Let $U \subset M$ be an open neighbourhood of image(z) such that $\overline{U}$ is compact. Set $V = M \backslash \overline{U}$.

   (a) Recall the LES for triples and apply it to $(M, U \cup V, V)$.

   (b) Use (a) to show $H_i(U) \cong H_i(U \cup V, V) = 0$ for $i > n$. Deduce that $a = 0$ in case $i > n$. *Hint:* Use the second part of the first lemma from lecture 12B, to see that some groups in the LES vanish.

   (c) To prove the theorem for $i = n$, consider the section $x \mapsto s(x) = L_{M,x}(a)$ of $\widetilde{M}_R \to M$. Show that $s = 0$. *Hint:* Note that it's enough to show $s(x_0) = 0$ for some $x_0 \in M$.

   (d) Deduce that $[z] = 0$ in $H_n(M|\overline{U}) = H_n(M, V)$. *Hint:* Apply the first part of the first lemma from lecture 12B.

   (e) Apply the LES in (a) to see that $[z] = 0 \in H_n(U)$ and deduce that $a = 0$.

8. Given two disjoint connected n-manifolds M_1 and M_2, their connected sum $M_1 \# M_2$ can be constructed by deleting the interiors of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. Show that for closed connected orientable n-manifolds M_1, M_2 there are isomorphisms

$$H_i(M_1) \oplus H_i(M_2) \cong H_i(M_1 \# M_2)$$

   for $0 < i < n$.

9. Show that if a closed orientable manifold of dimension $2n$ has $H_{n-1}(M)$ torsion-free then $H_n(M)$ is also torsion-free.

10. Compute the cup product structure of $H^*((S^2 \times S^8) \# (S^4 \times S^6))$, and in particular show that the only non-trivial cup products are those forced by Poincaré duality.

11. Show that if M is a compact connected non-orientable 3-manifold, $H_1(M)$ is infinite.

12. Prove that every map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has $\deg f = k^n$ for some $k \in \mathbb{Z}$.

13. Let $\alpha \in H^n(S^n)$ be a generator, and define $u = \alpha \times 1, v = 1 \times \alpha \in H^n(S^n \times S^n)$. Let now $f : S^n \times S^n \to S^n \times S^n$ be a map with $\deg f = \pm 1$. Writing $f^*(u) = au + bv$, $f^*v = cu + dv$ and assuming that n is even, prove that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

14. Let M be a closed connected orientable n-manifold and suppose that there exists a map $f : S^n \to M$ with $\deg f \neq 0$. Prove that $H_*(M;\mathbb{Q}) \cong H_*(S^n;\mathbb{Q})$. If $\deg f = \pm 1$, prove that $H_*(M;\mathbb{Z}) \cong H_*(S^n;\mathbb{Z})$.

15. Prove that if a closed connected orientable manifold M can be written as the union $M = U \cup V$ of two acyclic subsets, then $H_*(M) \cong H_*(S^n)$.