Solutions to problem set 3

Notation. We often omit the coefficient groups or rings from the notation, but they should always be clear from the context.

1. Consider the inclusions
$$i_\beta\colon (X_\beta, A_\beta) \to \left(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha\right).$$
They induce ring homomorphisms i_β^* in cohomology because the cup product in natural with respect to maps between spaces. By the UP of products of rings, we get a a unique ring homomorphism
$$\prod i_\alpha^*\colon H^*(\coprod X_\alpha, \coprod A_\alpha; R) \to \prod H^*(X_\alpha, A_\alpha; R).$$
It is left to show that this is an isomorphism. This follows from the UP:
$$\begin{array}{ccccccccc} 0 & \to & \prod \operatorname{Ext}(H_{n-1}(X_\alpha, A_\alpha), R) & \to & \prod H^n(X_\alpha, A_\alpha; R) & \to & \prod \operatorname{Hom}(H_n(X_\alpha, A_\alpha), R) & \to & 0 \\ & & \uparrow{\scriptstyle \prod (i_\alpha)_*^{\text{ext}}} & & \uparrow{\scriptstyle \prod i_\alpha^*} & & \uparrow{\scriptstyle \prod ((i_\alpha)_*)^*} & & \\ 0 & \to & \operatorname{Ext}(H_{n-1}(\coprod X_\alpha, \coprod A_\alpha), R) & \to & H^n(\coprod X_\alpha, \coprod A_\alpha); R) & \to & \operatorname{Hom}(H_n(\coprod X_\alpha, \coprod A_\alpha)), R) & \to & 0 \end{array}$$
The left and right vertical maps are isomorphisms because
$$\oplus(i_\alpha)_*\colon \bigoplus H_*(X_\alpha, A_\alpha) \to H_*(\coprod X_\alpha, \coprod A_\alpha)$$
is an isomorphism. By the 5-Lemma, the middle map is an isomorphism as well.
Note that $(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\})$ is a good pair. Therefore, the quotient map
$$q\colon (\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\}) \to (\bigvee_\alpha X_\alpha, *)$$
induces an isomorphism in homology:
$$q_*\colon H_*(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\}) \to \tilde{H}_*(\bigvee_\alpha X_\alpha).$$
Applying UCT and the five lemma we conclude that
$$q^*\colon \tilde{H}^*(\bigvee_\alpha X_\alpha; R) \to H^*(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\}; R)$$
is an isomorphism of groups. It is an isomorphism of rings because it is induced by q. Together with the first part, we proved the statement for the cohomology ring of a wedge product.

2. We have $H^n(X; R) \cong \operatorname{Hom}(H_n(X), R) \cong \operatorname{Hom}(H_n(X), \mathbb{Z}) \otimes R \cong H^n(X) \otimes R$ as Abelian groups, using the fact that $H_n(X)$ is free for all n and universal coefficient theorem for cohomology. Given a cocycle $\phi \in C^n(X)$ and $r \in R$, this isomorphism identifies the class $[\phi] \otimes r \in H^n(X) \otimes R$ with the class in $H^n(X; R)$ represented by the cocycle in $C^n(X; R)$ that takes a chain $\sigma \in C_n(X)$ to $\phi(\sigma)r$. That this respects the ring structures is immediate from the definitions.

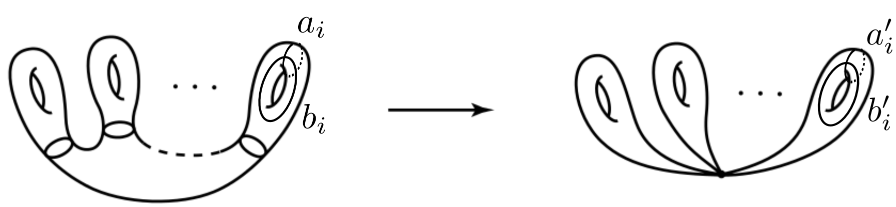
3. Consider the commutative diagram

$$\begin{array}{ccc} H^k(X,A) \times H^\ell(X,B) & \xrightarrow{\smile} & H^{k+\ell}(X, A \cup B) \\ \cong \downarrow & & \downarrow \\ H^k(X) \times H^\ell(X) & \xrightarrow{\smile} & H^{k+\ell}(X) \end{array}$$

The left vertical map is an isomorphism because A, B are acyclic and $k, \ell > 0$, as one sees by looking at the LES for the pairs (X, A) and (X, B); moreover, we have $H^{k+\ell}(X, A \cup B) = 0$ as $A \cup B = X$ by assumption. Combining these facts, it follows that the lower horizontal map vanishes.

If $X = A_1 \cup \cdots \cup A_n$ with acyclic open sets A_i, it follows in a similar way that all n-fold cup products of classes in $H^*(X)$ of positive dimensions vanish.

4. Denote by a_i, b_i, $i = 1, \ldots, g$, the standard basis elements of $H_1(\Sigma_g)$ and by a_i', b_i' the standard basis elements of $H_1(X)$, where $X = \bigvee_g T^2$ (as indicated in the figure). Moreover, let α_i, β_i be the elements of the dual basis of $H^1(\Sigma_g) \cong \mathrm{Hom}(H_1(\Sigma_g), \mathbb{Z})$, and α_i', β_i' the elements of the dual basis of $H^1(X) \cong \mathrm{Hom}(H_1(X), \mathbb{Z})$. We have $\pi_* a_i = a_i'$, $\pi_* b_i = b_i'$ as



$\pi : \Sigma_g \to X$ takes curves representing the classes on T^2 to curves representing the classes on X. Dualizing, it follows that $\pi^* \alpha_i' = \alpha_i$ and $\pi^* \beta_i' = \beta_i$.

The isomorphism $\iota_1^* \oplus \cdots \oplus \iota_g^* : H^*(X) \to \bigoplus_i H^*(T^2)$ induced by the inclusion maps $\iota_i : T^2 \to X$ is an isomorphism of rings, where the ring structure on the right is given by componentwise multiplication (see Problem 1). It follows that $\alpha_i' \smile \alpha_j' = \alpha_i' \smile \beta_j' = \beta_i' \smile \beta_j' = 0$ for $i \neq j$ because these classes live in different summands. Moreover, $\alpha_i' \smile \alpha_i' = 0 = \beta_i' \smile \beta_i'$ and $\alpha_i' \smile \beta_i' = (0, \ldots, 0, \gamma_{T^2}, 0, \ldots, 0) \in H^2(X)$ using that the cup product structure on $H^*(T^2)$ is known and denoting by γ_{T^2} a generator of $H^2(T^2)$ (for instance, $\iota_i^*(\alpha_i' \smile \beta_i') = \alpha \smile \beta = \gamma_{T^2} \in H^2(T^2)$ where now α, β denote generators of $H^1(T^2)$).

Denote by $[T^2]$ the generator of $H_2(T^2)$ dual to γ_{T^2} (note $H^2(T^2) \cong \mathrm{Hom}(H_2(T^2), \mathbb{Z})$) and by $[\Sigma_g]$ the generator of $H_2(\Sigma_g)$ such that $\pi_*([\Sigma_g]) = ([T^2], \ldots, [T^2])$ (one can see that such a generator exists using e.g. cellular homology). Then $(\alpha_i \smile \beta_i)[\Sigma_g] = (\pi^* \alpha_i' \smile \pi^* \beta_i')[\Sigma_g] = (\alpha_i' \smile \beta_i')(\pi_*[\Sigma_g]) = (\alpha_i' \smile \beta_i')([T^2], \ldots, [T^2]) = 1$, and hence $\alpha_i \smile \beta_i = \gamma_{\Sigma_g}$, the generator of $H^2(\Sigma_g) \cong \mathrm{Hom}(H_2(\Sigma_g), \mathbb{Z})$ dual to $[\Sigma_g]$; by skew-commutativity, we have $\beta_i \smile \alpha_i = -\alpha_i \smile \beta_i = -\gamma_{\Sigma_g}$. All other cup products between the basis elements of $H^1(\Sigma_g)$ vanish by the description above.

5. Let $\alpha \in C^k(A)$ and $\beta \in C^\ell(Y)$ be cocycles representing a and b. Recall that δa is represented by $\delta \overline{\alpha}$, where $\overline{\alpha} \in C^k(X)$ is any extension of α to a cochain in X and where the second δ is the coboundary homomorphism $C^*(X) \to C^{*+1}(X)$. Denote by $p_1 : (X \times Y, A \times Y) \to (X, A)$ and $p_2 : X \times Y \to Y$ the projections. With this notation, $\delta(a) \times b$ is represented by the relative cocycle $p_1^*(\delta\overline{\alpha}) \smile p_2^*(\beta)$. On the other hand, $\delta'(a \times b)$ is represented by the relative cocycle $\delta'(p_1^* \overline{\alpha} \smile p_2^* \beta) = p_1^*(\delta \overline{\alpha}) \smile p_2^*(\beta) \pm p_1^*(\overline{\alpha}) \smile p_2^*(\delta\beta) = p_1^*(\delta\overline{\alpha}) \smile p_2^*(\beta)$; here we use that $p_1^* \overline{\alpha} \smile p_2^* \beta \in C^{k+\ell}(X \times Y)$ is an extension of $p_1^* \alpha \smile p_2^* \beta \in C^{k+\ell}(A \times Y)$ and the fact that $\beta \in C^\ell(Y)$ is a cocycle.

6. Consider the LES in cohomology for the pair $(I \times Y, \partial I \times Y)$. Since the maps $i^* : H^n(I \times Y) \to H^n(Y \times \partial I)$ are injective (given by $i^*(a) = (a, a)$ in the obvious identifications $H^*(I \times Y) \cong H^*(Y)$ and $H^*(\partial I \times Y) \cong H^*(Y) \oplus H^*(Y)$), the LES splits into SESs of the form

$$0 \to H^n(I \times Y) \xrightarrow{i^*} H^n(\partial I \times Y) \xrightarrow{\delta'} H^{n+1}(I \times Y, \partial I \times Y) \to 0$$

which split as i^* has a left inverse (e.g. $(a, b) \mapsto a$). Define $1_0 \in H^0(\partial I)$ to be the class represented by the cocycle φ_0 with $\varphi_0(0) = 1$ and $\varphi_0(1) = 0$, and similarly define $1_1 \in H^0(\partial I)$. One checks easily that the composition $H^n(Y) \cong H^n(I \times Y) \xrightarrow{i^*} H^n(\partial I \times Y)$ is given by $b \mapsto 1_0 \times b + 1_1 \times b$, so the subspace $Q := \{1_0 \times b \mid b \in H^n(Y)\} \subset H^n(\partial I \times Y)$ is complementary to the image of i^*. Hence $\delta'|_Q : Q \to H^{n+1}(I \times Y, \partial I \times Y)$ is an isomorphism; since by the previous problem we have $\delta'(1_0 \times b) = \delta(1_0) \times b$, it follows that $H^n(Y) \to H^{n+1}(I \times Y, \partial I \times Y)$, $b \mapsto \delta(1_0) \times b$, is an isomorphism. This is what we need to prove in case $\mu_0 = \delta(1_0)$; any other generator $\mu_0 \in H^1(I, \partial I)$ is of the form $\mu_0 = \delta(1_0) \cdot r$ for some invertible $r \in R$, and thus $b \mapsto \mu_0 \times b$ is also an isomorphism in this case.

7. The $\mathbb{Z}_2$-cohomology ring of $\mathbb{R}P^3$ is $H^*(\mathbb{R}P^3; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^4)$ with $|\alpha| = 1$, whereas that of $\mathbb{R}P^2 \vee S^3$ is $H^*(\mathbb{R}P^2 \vee S^3; \mathbb{Z}_2) \cong \mathbb{Z}_2[\beta, \gamma]/(\beta^3, \gamma^2, \beta\gamma)$ with $|\beta| = 1$ and $|\gamma| = 3$ using the result of the Problem 1. These are isomorphic as $\mathbb{Z}_2$-vector spaces but not as rings (e.g. because $\alpha^3 \neq 0$, but the element of degree 1 in the second ring, β, satisfies $\beta^3 = 0$).

8. Using cellular homology, one computes

$$H_i(X; \mathbb{Z}), H_i(Y; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}_p & i = 2 \\ 0 & i = 3 \\ \mathbb{Z} & i = 4 \end{cases}$$

Using the universal coefficients theorem for cohomology it follows that

$$H^i(X; \mathbb{Z}), H^i(Y; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ 0 & i = 2 \\ \mathbb{Z}_p & i = 3 \\ \mathbb{Z} & i = 4 \end{cases} \quad \text{and} \quad H^i(X; \mathbb{Z}_p), H^i(Y; \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}_p & i = 2 \\ \mathbb{Z}_p & i = 3 \\ \mathbb{Z}_p & i = 4 \end{cases}$$

The cohomology rings with $\mathbb{Z}$-coefficients are clearly isomorphic (the only non-vanishing cup products are multiplication with multiples of the unit in H^0).

We now compare the cohomology rings with $\mathbb{Z}_p$-coefficients: Let $\alpha \in H^2(X; \mathbb{Z}_p)$ be a generator; using the cellular description of induced maps, one sees that the map $i^* : H^2(X; \mathbb{Z}_p) \to H^2(\mathbb{C}P^2; \mathbb{Z}_p)$ induced by the inclusion $i : \mathbb{C}P^2 \to X$ takes α to a generator $i^*(\alpha)$ of $H^2(\mathbb{C}P^2; \mathbb{Z}_p)$. Using the known description of the ring structure of $H^*(\mathbb{C}P^2)$ and the fact that $H^*(\mathbb{C}P^2; \mathbb{Z}_p) \cong H^*(\mathbb{C}P^2; \mathbb{Z}) \otimes \mathbb{Z}_p$ as rings (see Problem 2), it follows that $i^*(\alpha \smile \alpha) = i^*(\alpha) \smile i^*(\alpha) \neq 0$.

In contrast to that, if $\beta \in H^2(Y; \mathbb{Z}_p)$ is a generator, we have $\beta \smile \beta = 0$. To see that, recall the ring isomorphism $\widetilde{H}^*(Y; \mathbb{Z}_p) \cong \widetilde{H}^*(M(\mathbb{Z}_p; 2); \mathbb{Z}_p) \oplus \widetilde{H}^*(S^4; \mathbb{Z}_p)$ from Problem 1. The class β lives in the first factor of this splitting which vanishes in dimension 4.

9. We compute the differential of the cellular chain complex:

$$\begin{aligned}
&\partial x = \partial y = 0, \\
&\partial a = \partial b = y - x,\ \partial c = 0, \\
&\partial L = a - b + c,\ \partial U = -a + b + c.
\end{aligned}$$

It's straight forward to check that i is a chain map (basically because the singular differential satisfies the same formulas). The homology of the cellular chain complex is

$$H_0^{\mathrm{CW}}(\mathbb{R}P^2) = \mathbb{Z}[x],\ H_1^{\mathrm{CW}}(\mathbb{R}P^2) = \mathbb{Z}_2[a-b],\ H_2^{\mathrm{CW}}(\mathbb{R}P^2) = 0.$$

i_* is an isomorphism on H_0^{CW} because x also represents a generator of singular homology. i sends $a - b$ to a loop that generates $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ and hence by Hurewicz $i(a-b)$ is a generator of $H_1(\mathbb{R}P^2)$. This shows that i_* is an isomorphism on $H_1^{\mathrm{CW}}(\mathbb{R}P^2)$. All other homology groups vanish and so $i_*\colon H_*^{\mathrm{CW}}(\mathbb{R}P^2) \to H_*(\mathbb{R}P^2)$ is an isomorphism. Front and back faces of the cells are again cells. So we can apply the procedure from the lecture to compute the cup product on $H_*(\mathbb{R}P^2;\mathbb{Z}_2)$.
Denote by $x^*, y^*, a^*, b^*, c^*, U^*, L^* \in C^{\cdot}_{\mathrm{CW}}(\mathbb{R}P^2;\mathbb{Z}_2)$ the dual basis of $x, y, a, b, c, U, L \in C_{\cdot}^{\mathrm{CW}}(\mathbb{R}P^2;\mathbb{Z}_2)$. We compute

$$\begin{aligned}
&\delta x^* = \delta y^* = a + b^*, \\
&\delta a^* = \delta b^* = \delta c^* = U^* + L^*
\end{aligned}$$

and

$$H^0_{\mathrm{CW}}(\mathbb{R}P^2;\mathbb{Z}_2) = \mathbb{Z}_2[x^* + y^*],\ H^1_{\mathrm{CW}}(\mathbb{R}P^2;\mathbb{Z}_2) = \mathbb{Z}_2\alpha,\ H^2_{\mathrm{CW}}(\mathbb{R}P^2;\mathbb{Z}_2) = \mathbb{Z}_2[U^*],$$

where $\varphi = a^* + c^*$ and $\alpha = [\varphi]$. We compute $\alpha \cup \alpha$ (using Alexander-Whitney diagonal approximation for the cellular chains):

$$\begin{aligned}
\langle \varphi \cup \varphi, U \rangle &= (\varphi \otimes \varphi)(\Delta U) \\
&= (\varphi \otimes \varphi)([x_0, x_2] \otimes [x_2, x_3]) \\
&= \varphi \otimes \varphi)(c \otimes b) \\
&= (a^* + c^*)(c)(a^* + c^*)(b) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\langle \varphi \cup \varphi, L \rangle &= (\varphi \otimes \varphi)(\Delta U) \\
&= (\varphi \otimes \varphi)([x_0, x_2] \otimes [x_2, x_1]) \\
&= (\varphi \otimes \varphi)(c \otimes a) \\
&= (a^* + c^*)(c)(a^* + c^*)(a) \\
&= 1.
\end{aligned}$$

We conclude $\varphi \cup \varphi = L^*$ and $\alpha \cup \alpha = [L^*] = [U^*]$. A similar calculation shows that $x^* + y^*$ is a unit : $\alpha \cup (x^* + y^*) = \varphi$. We conclude that

$$H^*(\mathbb{R}P^2;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/\langle \alpha^3 = 0 \rangle$$

as rings.

10. (a) We see that $H_0^{\mathrm{CW}}(X) \cong \mathbb{Z}[x]$ and clearly $i(x) \in S_0(X)$ is a represents a generator $H_0(X) \cong \mathbb{Z}$. Hence i_* is an isomorphism on H_0.

    (b) $H_1^{\mathrm{CW}}(X) \cong \mathbb{Z}[a] \oplus \mathbf{Z}[b]$. a and b are generators of $\pi_1(X) \cong \mathbb{Z} \otimes \mathbb{Z}$. Therefore, $i(a)$ and $i(b)$ are a basis of $H_1(X) \cong \mathbb{Z} \otimes \mathbb{Z}$ by Hurewicz.

    (c) i. The cellular differential satisfies $\partial^{\mathrm{CW}}(L) = \partial^{\mathrm{CW}}(U) = a + b - c$. L and U are the only 2-cells and so the cycles of degree 2 are precisely multiples of $L - U$. We conclude $H_2^{\mathrm{CW}}(X) = \mathbb{Z} \cdot [F]$.

    ii. Consider the following sequence of isomorphisms:

$$\begin{array}{ccccc} H_2(X,U) & \xleftarrow[\cong]{\mathrm{incl}_*} & H_2(X\backslash \mathrm{Int}(U), U\backslash \mathrm{Int}(U)) = H_2(L, a\cup b\cup c) & \xrightarrow[\cong]{q_*} & H_2(L/(a\cup b\cup c), *) \\ & & & & \downarrow \cong \\ H_1(\partial\Delta) & \xleftarrow[\cong]{\partial_*} & H_2(\Delta^2, \partial\Delta^2) & \xrightarrow[\cong]{p_*} & H_2(\Delta^2/\partial\Delta^2, *) \end{array}$$

    The first isomorphism comes from excision. q_* is an isomorphism because $(L, a \cup b \cup c)$ is a good pair (q is the obvious projection from L to the quotient). Similary, p_* is an isomorphism. ∂_* is an isomorphism coming from the LES for the pair $(\Delta^2, \partial\Delta^2)$. Under these isomorphisms, $[j_*(F')]$ is sent to the generator $\partial_*([\Delta]) = [\partial\Delta] \in H_1(\partial\Delta^1) \cong \mathbb{Z}$. Hence, $[j_*(F')] \in H_2(X,U)$ generates $H_2(X,U) \cong \mathbb{Z}$.

    iii. Since $j_*([F'])$ is a generator, also $[F'] \in H_2(X) \cong \mathbb{Z}$ is a generator. Thus $i_* \colon H_2^{\mathrm{CW}}(X) \to H_2(X)$ sends the generator $[F]$ to a generator. Thus i_* is an isomorphism on H_2.

11. (a) See page 7 in the lecture notes from 28.4.21

    (b) We omit the ring R in the notation. Let $\alpha \in H^*(A)$ and $\beta \in H^*(X)$. Let $\varphi \in S^p(A)$ representing α. Extend φ to a cochain $\hat{\varphi} \in S^p(X)$ with $\hat{\varphi}|_{S_p(Y)} = \varphi$. Let $\psi \in S^q(X)$ be a cocycle with $[\psi] = \beta$. Then $\alpha \cup i^*\beta$ is represented by

$$\varphi \cup (\psi|_{S_q(A)}) = (\hat{\varphi} \cup \psi)|_{S_{p+q}(A)}.$$

    So $\hat{\varphi} \cup \psi$ is an extension to $S_{p+q}(X)$ of $\varphi \cup (\psi|_{S_q(A)})$. We compute:

$$\begin{aligned} \delta^*(\alpha \cup i^*\beta) &= [\delta(\hat{\varphi} \cup \psi)] \\ &= [\delta\hat{\varphi} \cup \psi + (-1)^p \hat{\varphi} \cup \delta\psi] \\ &= [\delta\hat{\varphi} \cup \psi] \\ &= \delta^*(\alpha) \cup \beta. \end{aligned}$$

    The third equality uses that ψ is a cocycle. The equality

$$\delta^*(i^*\beta \cup \alpha) = \beta \cup \delta^*(\alpha)$$

    is proved similarly. The other formulas involving j^* follow directly from the definition of the relative cup product.
    Remark: This shows that all the maps in the LES of (X, A) are maps of $H_*(X)$-modules.

12. First, let X, Y be any spaces and R a commutative ring. Consider the cohomological cross product

$$\begin{aligned} H^*(X;R) \otimes_R H^*(Y;R) &\longrightarrow H^*(X \times Y; R) \\ \alpha \otimes \beta &\longmapsto \alpha \times \beta. \end{aligned}$$

Property 4) of the cup product in lecture 9A states that

$$(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2) = (-1)^{|\beta_1||\alpha_2|}(\alpha_1 \cup \alpha_2) \times (\beta_1 \cup \beta_2)$$

for any $\alpha_1, \alpha_2 \in H^*(X; R)$ and $\beta_1, \beta_2 \in H^*(Y; R)$. Applying the universal coefficients theorem and the Künneth formula in homology, we see that

$$H^*(T^n; \mathbb{Z}) \cong H^*(S^1; \mathbb{Z}) \otimes \cdots \otimes H^*(S^1; \mathbb{Z}).$$

(This uses that all homology groups involved are finitely generated **free** abelian groups, so that all Tor and Ext- groups vanish.) In fact, this isomorphism is precisely the cohomological cross product (see Theorem 3.2. in Chapter VI.3. in Bredon). By the calculation above, this isomorphism is an isomorphism of graded rings. (The product on $A \otimes B$ for graded rings A and B is defined by $(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb'$.) We know the homology of the sphere:

$$H^*(S^1; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/\langle \alpha \cdot \alpha = 0 \rangle.$$

We conclude:

$$\begin{aligned} H^*(T^n; \mathbb{Z}) &\cong H^*(S^1; \mathbb{Z}) \otimes \cdots \otimes H^*(S^1; \mathbb{Z}) \\ &\cong \mathbb{Z}[\alpha_1, \ldots, \alpha_n]/\langle \alpha_i \cdot \alpha_i = 0, \alpha_i \cdot \alpha_j = -\alpha_j \cdot \alpha_i \rangle \\ &\cong \Lambda_{\mathbb{Z}}[\alpha_1, \cdots, \alpha_n]. \end{aligned}$$