Prof. Paul Biran
ETH Zürich

Algebraic Topology II

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Solutions to problem set 4

1. Let N be an orientable manifold and let $f : M \to N$ be a covering map. Then M is also a manifold as f is a local homeomorphism. We now explain how to construct an orientation of M from a given orientation of N. To do so, let $x \in M$ and fix an open ball $B \ni x$ which gets mapped by f homeomorphically onto $f(B)$. This induces an identification $H_n(M|x) \cong H_n(N|f(x))$ using the isomorphisms

$$H_n(M|x) \leftarrow H_n(B|x) \to H_n(f(B)|f(x)) \to H_n(N|f(x)),$$

and hence a given local orientation $\mu_{f(x)} \in H_n(N|f(x))$ at $f(x)$ induces a local orientation $\mu_x \in H_n(M|x)$ at x. (This is in fact independent of the chosen ball: Given two balls B_0, B_1 around x as above, choose another ball $B_{01} \subset B_0 \cap B_1$ around x and consider the commutative diagram obtained via the inclusions $B_{01} \to B_i \to M$ and $f(B_{01}) \to f(B_i) \to N$...) It remains to show that these local orientations satisfy the local consistency condition required in the definition of an orientation. Choose another ball C around x such that $\overline{C} \subset B$ and choose a generator $\mu_{f(C)} \in H_n(N|f(C))$ such that for every point $y \in C$, the map $H_n(N|f(C)) \to H_n(N|f(y))$ takes $\mu_{f(C)}$ to the $\mu_{f(y)}$, the local orientation at $f(y)$ determined by the chosen orientation of N. Consider now the commutative diagram

$$\begin{array}{ccccccc} H_n(M|C) & \longleftarrow & H_n(B|C) & \longrightarrow & H_n(f(B)|f(C)) & \longrightarrow & H_n(N|f(C)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_n(M|y) & \longleftarrow & H_n(B|y) & \longrightarrow & H_n(f(B)|f(y)) & \longrightarrow & H_n(N|f(y)) \end{array}$$

Define $\mu_C \in H_n(B|C)$ to be the generator obtained from $\mu_{f(C)} \in H_n(N|f(C))$ via the isomorphisms in the top row (which is independent of y). Note that the generator of $H_n(M|y)$ we obtain from $\mu_{f(y)} \in H_n(N|f(y))$ via the isomorphisms in the bottom row is precisely μ_y (as indicated above, any sufficiently small ball around y can be used to define this generator). Since the right vertical map takes $\mu_{f(C)} \mapsto \mu_{f(y)}$, these facts together with the commutativity of the diagram imply that the left vertical map takes $\mu_C \mapsto \mu_y$. This proves that the local consistency condition holds, and hence the local orientations on M obtained from those on N fit together to give an orientation.

2. We have seen in class that every manifold M has an orientable double cover, namely $\widetilde{p} : \widetilde{M} \to M$ with $\widetilde{M} = \{(x, \mu_x) \mid \mu_x \text{ generates } H_n(M|x)\}$. Hence what we have to show is that if M is non-orientable, then this is unique up to isomorphism.

So let $\overline{p} : \overline{M} \to M$ be another orientable double cover. Note first that $\overline{M}$ is connected, because if it were disconnected, it would have two components each of which would be homeomorphic to M. But then an orientation of $\overline{M}$ would yield an orientation of M, contradicting the non-orientability of the latter.

Fix now an orientation on $\widetilde{M}$, given by a choice of local orientation $\mu_{\overline{x}} \in H_n(\overline{M}|\overline{x})$ at every $\overline{x} \in \overline{M}$. We now define a map $\phi : \overline{M} \to \widetilde{M}$ as follows: Given $\overline{x} \in \overline{M}$, denote by $\overline{p}_* \mu_{\overline{x}} \in H_n(M|x)$ the local orientation at $x = \overline{p}(\overline{x})$ induced by $\mu_{\overline{x}}$ (using a small ball around $\overline{x}$ on which $\overline{p}$ restricts to a homeomorphism). Then define

$$\phi : \overline{M} \to \widetilde{M}, \quad \overline{x} \mapsto (\overline{p}(\overline{x}), \overline{p}_* \mu_{\overline{x}}).$$

ϕ clearly a local homeomorphism, in particular a continuous map. Moreover, it satisfies $\widetilde{p} \circ \phi = \overline{p}$, and hence $\overline{p}_* \pi_1(\overline{M}) \subset \widetilde{p}_* \pi_1(\widetilde{M})$. Since both are index 2 subgroups of $\pi_1(M)$, it follows that $\overline{p}_* \pi_1(\overline{M}) = \widetilde{p}_* \pi_1(\widetilde{M})$. But we know from the theory of covering spaces that this group determines a connected cover uniquely up to isomorphism, so $\overline{M}$ and $\widetilde{M}$ are isomorphic.

3. We claim that for any ball $B \subset M$ the map $f : M \to M/(M \setminus B) \approx S^n$ given by collapsing all of $M \setminus B$ to a point has degree ± 1. To see this, take a point $x \in B$ and consider the commutative diagram

$$\begin{array}{ccc} H_n(M) & \longrightarrow & H_n(S^n) \\ \downarrow & & \downarrow \\ H_n(M|x) & \longrightarrow & H_n(S^n|f(x)) \\ \uparrow & & \uparrow \\ H_n(B|x) & \longrightarrow & H_n(f(B)|f(x)) \end{array}$$

Here the vertical maps on the left are induced by the inclusions $M \to (M, M \setminus x)$ resp. $(B, B \setminus x) \to (M, M \setminus x)$, and similarly for ones on the right; the horizontal maps are induced by f. Note that the vertical maps are isomorphisms (the upper ones by orientability, the lower ones by excision), and so is the lower horizontal map because $f : (B, B \setminus x) \to (f(B), f(B) \setminus f(x))$ is a homeomorphism. It follows from the commutativity of the diagram that also the upper horizontal map is an isomorphism, which is equivalent to saying that the degree of f is ± 1.

4. Choose some $x \in B$ and denote by $x_i \in B_i$ its preimage under $f|_{B_i}$. Consider the commutative diagram

$$\begin{array}{ccccc} H_n(M) & \longrightarrow & H_n(M|x_1, \ldots, x_n) & \xrightarrow{\cong} & \bigoplus_i H_n(B_i|x_i) \\ \downarrow & & \downarrow & & \downarrow \\ H_n(N) & \xrightarrow{\cong} & H_n(N|x) & \xrightarrow{\cong} & H_n(B|x) \end{array}$$

in which the vertical maps are induced by f, the left horizontal ones by inclusion, and where the right horizontal isomorphisms come from excision of $M \setminus (B_1 \cup \cdots \cup B_n)$ resp. $N \setminus B$. Denote by $\mu_{x_i} \in H_n(B_i|x_i)$ resp. $\mu_x \in H_n(B|x)$ generators corresponding to the fundamental classes $[M]$ and $[N]$. Then $[M]$ gets mapped to $(\mu_{x_1}, \ldots, \mu_{x_k}) \in \bigoplus_i H_n(B_i|x_i)$ under the composition of the top horizontal maps which in turn gets mapped to $\sum_i \varepsilon_i \mu_x \in H_n(B|x)$ by the right vertical map (by our assumption on the $f|_{B_i} : B_i \to B$). On the other hand, the left vertical map takes $[M]$ to $\deg(f)[N]$, which then is mapped to $\deg(f)\mu_x \in H_n(B|x)$ by the composition of the bottom horizontal maps. Commutativity of the diagram yields $\deg(f) = \sum_i \varepsilon_i$.

5. Let $x \in N$ and let $B \ni x$ be a ball such that $f^{-1}(B)$ is a disjoint union of balls $B_1, \ldots, B_p$ which f maps homeomorphically onto B and denote by $x_i \in B_i$ the preimages of x. In view of Problem 4, we only have to show that the local contributions to the degree corresponding to each B_i are the same. Suppose that an orientation of N has been fixed, which yields a local orientation μ_x at x. As we have seen in Problem 1, the local orientations μ_{x_i}, $i = 1, \ldots, p$, obtained from μ_x via the identifications $f_* : H_n(B_i|x_i) \to H_n(B|x)$ come from an orientation of M, namely the one obtained by similarly pulling back local orientations from N at all points of M (we have shown there that these local orientations fit together to an orientation).

Turning this statement upside down, we see that this orientation induces local orientations at $x_1, \ldots, x_p$ which all get mapped to the same local orientation at x, so all contribute with the same sign.

6. [See p.152 of Hatcher's book]

   Note that all columns and the first two rows of the diagram are exact. Using this information, it is easy to see that the third row is a chain complex (i.e. the two maps compose to zero). Moreover, the diagram is a SES of the chain complexes given by the rows. The corresponding LES in homology allows to conclude that the third row is exact (which is equivalent to saying it has vanishing homology) because that's true for the first two rows. Since the horizontal maps are clearly chain maps, we therefore have a SES of chain complexes

$$0 \to S_*(Q \cap R, S \cap T) \to S_*(Q, S) \oplus S_*(R, T) \to S_*(Q + R, S + T) \to 0 \tag{1}$$

   We will show that the homology of the last term is isomorphic to $H_n(X, Y)$. Taking this into account, the LES corresponding to the SES of chain complexes (1) is as desired.

   Consider the third column in the commutative diagram for which there is an obvious map of SES as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_*(S+T) & \longrightarrow & S_*(Q+R) & \longrightarrow & S_*(Q+R, S+T) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_*(Y) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X, Y) & \longrightarrow & 0 \end{array}$$

   Consider now the resulting commutative diagram in homology and recall from class that the first two vertical maps induce isomorphisms in homology. Using the 5-lemma, we deduce that also $S_*(Q + R, S + T) \to S_*(X, Y)$ induces an isomorphism in homology.

7. (a) Consider a triple (X, A, B) with $B \subset A \subset X$. The short exact sequence

$$0 \to S(A, B) \to S(X, B) \to S(X, A) \to 0$$

   induces the long exact sequence

$$\cdots \to H_{i+1}(X, A) \to H_i(A, B) \to H_i(X, B) \to H_i(X, A) \to \cdots$$

   Applied to the triple $(M, U \cup V, V)$ we get an exact sequence

$$\cdots \to H_{i+1}(M, U \cup V) \to H_i(U \cup V, V) \to H_i(M, V) \to H_i(M, U \cup V) \to \cdots$$

   (b) For $i > n$, $H_{i+1}(M, U \cup V) = H_{i+1}(M|M \backslash (U \cup V)) = 0$ by the lemma from lecture 12B and because $M \backslash (U \cup V) = 0$ is compact. For the same reason, $H_i(M, V) = H_i(M|\overline{U}) = 0$. From the LES we deduce $H_i(U) \cong H_i(U \cup V, V) = 0$. In particular, $[z] = 0 \in H_i(U)$ and so $z = \partial w$ for some chain $w \subset U \subset M$. It follows that also $[z] = 0 \in H_i(M)$.

   (c) Since M is connected, s is uniquely determined by its value at some point $x_0 \in M$. So $s \equiv 0$ if and only if $s(x_0) = 0$. Since image(z) is compact and M is not, there exists $x_0 \in M \backslash \text{image}(z)$. We have $s(x_0) = L_{M,x_0}([z]) = 0$ and thus $s \equiv 0$.

   (d) By the lemma, there is a unique $\alpha \in H_n(M|\overline{U})$ with $L_x(\alpha) = s(x)$. But both, $0 \in H_n(M|\overline{U})$ and $L_{M,\overline{U}}(a)$ solve the equation. Hence $L_{M,\overline{U}}(a) = 0$, which means that $[z] = 0 \in H_n(M|\overline{U}) = H_n(M, V)$.

(e) The LES in degree n reads

$$0 = H_{n+1}(M, U \cup V) \to H_n(U \cup V, V) \xrightarrow{j} H_n(M, V) \to \cdots$$

$[z]$ viewed as a class in $H_n(U) \cong H_n(U \cup V, V)$ is sent to $[z] = 0 \in H_n(M, V)$ via j. j is injective by exactness of the sequence and hence $[z] = 0 \in H_n(U)$. Write $z = \partial w$ for some $w \in S_{n+1}(U)$ to conclude $[z] = 0 \in H_n(M)$.

8. Given a closed manifold M, we set $M' := M \setminus \text{pt}$. Using the Mayer-Vietoris sequence for the cover of M given by M' and a ball, one sees that $H_i(M') = H_i(M)$ for $i < n-1$. We also know that $H_n(M') = 0$ as M' is non-compact, and hence the top end of the MV sequence is

$$0 \to H_n(M) \to H_{n-1}(S^{n-1}) \to H_{n-1}(M') \to H_{n-1}(M) \to 0 \tag{2}$$

Since M is orientable ($\Leftrightarrow H_n(M) \cong \mathbb{Z}$), the first map is an isomorphism so that we get $H_{n-1}(M') = H_{n-1}(M)$.

To compute $H_*(M_1 \# M_2)$, consider the cover of $M_1 \# M_2$ given by two sets $A_1 \approx M_1'$, $A_2 \approx M_2'$ with $A_1 \cap A_2 \simeq S^{n-1}$. From the resulting Mayer-Vietoris sequence we see immediately that $H_i(M_1 \# M_2) \cong H_i(M_1') \oplus H_i(M_2') \cong H_i(M_1) \oplus H_i(M_2)$ for $0 < i < n-1$. The top end of the MV sequence looks as follows:

$$0 \to H_n(M_1 \# M_2) \to H_{n-1}(S^{n-1}) \xrightarrow{\phi} H_{n-1}(M_1') \oplus H_{n-1}(M_2') \to H_{n-1}(M_1 \# M_2) \to 0 \tag{3}$$

Writing $\phi = (\phi_1, \phi_2)$, note that the maps $\phi_i : H_{n-1}(S^{n-1}) \to H_{n-1}(M_i')$ are precisely those also appearing in (2) with $M = M_i$. So since both M_1 and M_2 are orientable, ϕ vanishes and we obtain $H_{n-1}(M_1 \# M_2) \cong H_{n-1}(M_1) \oplus H_{n-1}(M_2)$; we also see that $H_n(M_1 \# M_2) \cong \mathbb{Z}$, so $M_1 \# M_2$ is orientable.

9. Poincaré duality tells us that $H_{n-1}(M) \cong H^{n+1}(M)$, and $H^{n+1}(M) \cong \text{Hom}(H_{n+1}, \mathbb{Z}) \oplus \text{Ext}(H_n(M), Z)$ by the universal coefficient theorem. Recall that $\text{Ext}(G, \mathbb{Z})$ is isomorphic to the torsion subgroup of G for any finitely generated Abelian group G (which the $H_i(M)$ are as M is a compact manifold). So if $H_n(M)$ has torsion, then also $H^{n+1}(M)$ and $H_{n-1}(M)$ have torsion.

10. Consider the maps $p : (S^2 \times S^8) \# (S^4 \times S^6) \to S^2 \times S^8$ and $q : (S^2 \times S^8) \# (S^4 \times S^6) \to S^4 \times S^6$ given by collapsing one of the two summands. It follows from the result of problem 1 that $p^* \oplus q^* : H^i(S^2 \times S^8) \oplus H^i(S^4 \times S^6) \to H^i(S^2 \times S^8 \# S^4 \times S^6)$ is an isomorphism in degrees $0 < i < 10$ (check that our $p^* \oplus q^*$ is just the dual of the isomorphism $H_i(S^2 \times S^8 \# S^4 \times S^6) \cong H_i(S^4 \times S^6) \oplus H_i(S^2 \# S^8)$ from problem 1!). The fact that this is a ring homomorphism shows that the only non-trivial cup products are those between elements of complementary degrees, i.e. those forced by Poincaré duality.

11. Recall from class that every odd-dimensional manifold has vanishing Euler characteristic, so $0 = \chi(M) = b_0 - b_1 + b_2 - b_3$. We have $b_0 = 1$, and $b_3 = 0$ since M is non-orientable. Hence $b_1 > 0$ and thus $H_1(M)$ is infinite.

12. Denoting by $\alpha \in H^2(\mathbb{C}P^n)$ a generator, α^n generates $H^{2n}(\mathbb{C}P^n)$ and we have $\alpha^n \frown [\mathbb{C}P^n] = 1$ for one choice of fundamental class $[\mathbb{C}P^n] \in H_{2n}(\mathbb{C}P^n)$. Take $k \in \mathbb{Z}$ such that $f^*(\alpha) = k\alpha$. Then $f^*(\alpha^n) = (f^*(\alpha))^n = k^n \alpha^n$ and hence $\alpha^n \frown f_*([\mathbb{C}P^n]) = f^*(\alpha^n) \frown [\mathbb{C}P^n] = k^n \alpha^n \frown [\mathbb{C}P^n] = k^n$. It follows that $f_*([\mathbb{C}P^n]) = k^n [\mathbb{C}P^n]$, i.e. f has degree k^n.

13. The map $H^n(S^n) \oplus H^n(S^n) \cong H^n(S^n \times S^n)$, $(k\alpha, \ell\alpha) \mapsto k(\alpha \times 1) + \ell(1 \times \alpha) = ku + \ell v$, is an isomorphism by Künneth. Being a product of orientable manifolds, $S^n \times S^n$ is orientable and thus the cup product pairing is non-singular, which implies that there exists some $u' \in H^n(S^n \times S^n)$ such that $u \smile u'$ generates $H^{2n}(S^n \times S^n)$. Since $u \smile u = (p_0^*\alpha \smile p_1^*1) \smile (p_0^*\alpha \smile p_1^*1) = p_0^*(\alpha \smile \alpha) \smile p_1^*(1 \smile 1) = 0$ as $\alpha \smile \alpha = 0$ (where $p_i : S^n \times S^n \to S^n$ denote the projections to the factors), we can choose $u' = v$.

    So $u \smile v$ generates $H^{2n}(S^n \times S^n)$, and thus by Poincaré duality we know that $(u \smile v) \frown [S^n \times S^n] = \pm 1$, where $[S^n \times S^n]$ is a fundamental class. It follows that $f^*(u \smile v) = \pm u \smile v$, using that $f^*(u \smile v) \frown [S^n \times S^n] = (u \smile v) \frown f_*[S^n \times S^n] = \pm 1$ by the assumption that $\deg f = \pm 1$. Note that $u \smile v = v \smile u$ a n is even; using that and $u \smile u = 0 = v \smile v$, we obtain

    $$f^*(u \smile v) = f^*(u) \smile f^*(v) = (au + bv) \smile (cu + dv) = (ad + bc) u \smile v,$$

    $$f^*(u \smile u) = (au + bv) \smile (au + bv) = 2ab(u \smile v) = 0,$$
    $$f^*(v \smile v) = (cu + dv) \smile (cu + dv) = 2cd(u \smile v) = 0.$$

    So $ad + bc = \pm 1$ and $ab = 0 = cd$, which is equivalent to what we need to prove.

14. Let $[S^n] \in H_n(S^n; \mathbb{Q})$ and $[M] \in H_n(M; \mathbb{Q})$ be fundamental classes. Writing $k = \deg f$, we have $f_*[S^n] = k[M]$. Fix now $0 < i < n$ and let $\sigma \in H_i(M; \mathbb{Q})$ be any class with Poincaré dual $\alpha \in H^{n-i}(M; \mathbb{Q})$, i.e. $\sigma = \alpha \frown [M]$. Then $k\sigma = k\alpha \frown [M] = \alpha \frown f_*[S^n] = f^*\alpha \frown [S^n] = 0$ because $f^*\alpha \in H^{n-i}(S^n)$ vanishes for degree reasons. Since we are working over $\mathbb{Q}$, it follows that $\sigma = 0$. So $H_i(M; \mathbb{Q}) = 0$ for $0 < i < n$ and we conclude $H_*(M; \mathbb{Q}) \cong H_*(S^n; \mathbb{Q})$ (in degrees 0 and n this is clear as M is closed connected orientable). If we replace $\mathbb{Q}$ by $\mathbb{Z}$, the same argument shows that every $\sigma \in H_k(M)$ is k-torsion; in particular, if we assume $k = \pm 1$ we obtain $H_i(M) = 0$ for $0 < i < n$ and thus $H_*(M) \cong H_*(S^n)$.

15. Suppose that $H_i(M) \neq 0$ for some $0 < i < n$. If $H_i(M)$ contains a non-torsion element σ, consider its Poincaré dual $\alpha \in H^{n-i}(M)$; by the non-singularity of the cup product pairing there exists some $\beta \in H^i(M)$ such that $\alpha \smile \beta$ generates $H^n(M)$. Otherwise there is a non-zero σ that is p-torsion for some prime p and the universal coefficient theorem for homology implies that there exists a non-zero element $\sigma' \in H_i(M; \mathbb{Z}_p)$ with Poincaré dual $\alpha' \in H^{n-i}(M; \mathbb{Z}_p)$; by the non-singularity of the cup product pairing (now over the field $\mathbb{Z}_p$) there exists some $\beta' \in H^i(M; \mathbb{Z}_p)$ such that $\alpha' \smile \beta'$ generates $H^n(M; \mathbb{Z}_p) \cong \mathbb{Z}_p$.

    On the other hand, the assumption that $M = U \cup V$ with acyclic U, V implies that all cup products of classes of positive degree in $H^*(M)$ resp. $H^*(M; \mathbb{Z}_p)$ vanish.

    This contradiction shows that $H_i(M) = 0$ for $0 < i < n$ and hence $H_*(M) \cong H_*(S^n)$.