

## Problem set on tensor products

Let  $R$  be a commutative ring. Below we omit  $R$  from the notation  $\otimes_R$  whenever the ring  $R$  is obvious from the context. When no ring  $R$  is mentioned,  $\otimes$  means tensor product over  $\mathbb{Z}$ .

1. Let  $M$  be an  $R$ -module. Show that for every exact sequence of  $R$ -modules  $U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$  the sequence

$$M \otimes U \xrightarrow{\text{id} \otimes f} M \otimes V \xrightarrow{\text{id} \otimes g} M \otimes W \rightarrow 0$$

is exact. *Hint:* To prove exactness at  $M \otimes V$ , construct a left-inverse for an appropriate map  $M \otimes V / \text{im}(\text{id} \otimes f) \rightarrow M \otimes W$ .

2. Let  $0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$  be a short exact sequence of  $R$ -modules. Assume the sequence splits. Prove that for every  $R$ -module  $M$  the sequence

$$0 \rightarrow U \otimes M \xrightarrow{f \otimes \text{id}} V \otimes M \xrightarrow{g \otimes \text{id}} W \otimes M \rightarrow 0$$

is exact.

3. Let  $M$  be a free  $R$ -module. Show that for every short exact sequence  $0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$  the sequence

$$0 \rightarrow M \otimes U \xrightarrow{\text{id} \otimes f} M \otimes V \xrightarrow{\text{id} \otimes g} M \otimes W \rightarrow 0$$

is exact.

4. Find counterexamples to the statements in problems 2, resp. 3, in case we drop the assumption that the sequence  $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$  splits, resp. the module  $M$  is free.
5. Let  $M$  be an  $R$ -module and  $J \subset R$  an ideal. Consider the submodule of  $M$  generated by  $\{a \cdot m \mid a \in J, m \in M\}$  and denote this submodule by  $JM$ . Consider also  $R/J$  as an  $R$ -module. Prove that

$$(R/J) \otimes_R M \cong M/JM.$$

6. Prove that  $\mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_d$ , where  $d = \text{gcd}(n, m)$ .

7. An abelian group  $G$  is called *divisible* if for all  $g \in G, 0 \neq n \in \mathbb{Z}$ , there exists  $h \in G$  such that  $nh = g$ . For example,  $G$  is a field of characteristic 0, endowed with the  $+$  operation, or  $G = \mathbb{R}/\mathbb{Z}$ , or  $G = \mathbb{Q}/\mathbb{Z}$ .

An abelian group  $T$  is called *torsion* if for all  $t \in T$ , there exists  $0 \neq n \in \mathbb{Z}$  such that  $nt = 0$ . Prove that if  $G$  is divisible and  $T$  is torsion then  $G \otimes T = 0$ .

This can be generalized to  $R$ -modules: An  $R$ -module  $M$  is called *divisible* if for any  $m \in M$  and any  $r \in R$ , which is not a 0-divisor, there exists  $n \in M$  such that  $rn = m$ . An  $R$ -module  $T$  is called *torsion*, if for each  $m \in M$ , there exists  $r \in R$  which is not a zero-divisor, such that  $rm = 0$ . If  $M$  is divisible and  $T$  is torsion, then  $M \otimes_R T = 0$ .

8. Prove that there exists isomorphisms

$$\text{hom}_R(U, \text{hom}_R(V, W)) \cong \text{Bilin}_R(U, V; W) \cong \text{hom}_R(U \otimes_R V, W)$$

for all  $R$ -modules  $U, V, W$  that are natural in  $U$ , in  $V$  and in  $W$ .

9. Let  $U, V$  be free  $R$ -modules of finite rank. Consider  $U^* := \text{hom}_R(U, R)$  viewed as an  $R$ -module. Prove that there exists an isomorphism

$$U^* \otimes_R V \cong \text{hom}_R(U, V)$$

which is natural in  $U$  and in  $V$ .