Exercises: Week 2

Computation in Algebra and Arithmetic

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Sage provides standard constructions from linear algebra. A brief introduction on how matrices and vectors are implemented can be found here: `https://doc.sagemath.org/html/en/tutorial/tour_linalg.html`.

1 Working with matrices

Perform the following tasks to become more familiar with the implementation of linear algebra in Sage.

(1) Consider the matrix $A := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ over $\mathbf{Q}$. Compute its determinant, RREF and the related transformation matrix. What is the kernel of A?
Note: When we talk about the kernel of a matrix M, we refer to the kernel of the linear map $v \mapsto vM$. Sometimes this is also called the "left-kernel" of M, whereas the "right-kernel" of M is the kernel of $v \mapsto Mv$. In sage calling "M.left_kernel()" or just "M.kernel()" return the left-kernel and "M.right_kernel()" returns the right-kernel.

(2) Let B the image of A in $M_{4\times 4}(\mathbf{F}_2)$. Compute the left-kernel of B, the right-kernel of B and give their respective RREF bases.
Note: If p is a prime, the expression "GF(p)" references the field of size p in sage. Moreover, the method "change_ring()" might prevent you from having to type down the matrix again.

(3) Now consider the matrix A from (1) as a matrix over $\mathbf{Z}$. Compute both the Smith form of A as well as the Hermite form of A.

(4) Compute the Jordan form of A over $\bar{\mathbf{Q}}$.

2 Subspaces in sage

Note: You won't have to do a lot by hand. For the sage implementation of vectorspaces, see `https://doc.sagemath.org/html/en/constructions/linear_algebra.html`.

(1) Let W_1 be the subspace of $\mathbf{Q}^4$ generated by $[5, 7, \frac{3}{4}, 0]$ and $[1, 2, 3, 4]$. Similarly, let W_2 be the subspace of $\mathbf{Q}^4$ generated by $[0, 1, 0, 0]$. Compute a basis of $W_3 := W_1 + W_2$. Which of the following vectors are contained in W_3?

- $[1, 0, 1, \frac{68}{57}]$
- $[1, 0, 0, 0]$
- $[0, 0, 1, 0]$

(2) Let $v_1 = [1, 0, 0, 0]$, $v_2 = [0, 0, 1, 0]$, $v_3 = [0, 0, 0, 1]$, $w_1 = [0, 1, 0, 0]$ and $w_2 = [0, 0, 1, \frac{1}{2}]$. Moreover, let $V \subset \mathbf{Q}^4$ be the subspace spanned by the v_i and $W \subset \mathbf{Q}^4$ the subspace spanned by the w_i.

- Compute $W \cap V$.
  Hint: The method "intersection()" exists.
- What is the echelon basis of $W \cap V$?