Exercises: Week 5

Computation in Algebra and Arithmetic

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Sage provides standard constructions for polynomials. A brief introduction on how they are implemented can be found here: `https://doc.sagemath.org/html/en/tutorial/tour_polynomial.html`.

1 Graded lexicographic order

In the lecture, the standard lexicographic order was introduced, but this is by no means the only total order on the monomials. In this exercise, we want to explore the **degree lexicographic order**. This order, abbreviated with **deglex**, or **grlex** for *graded lexicographic order* first compares the total degree (sum of all exponents), and in case of a tie applies lexicographic order.

(1) Show that deglex is a total order on the monomials.

(2) Show that with respect to this order, any monomial is only preceded by a finite number of monomials.

Note: *By default, Sage actually uses neither deglex nor lex, but the* ***Graded reverse lexicographic order*** *(grevlex, or degrevlex for degree reverse lexicographic order), which compares the total degree first, then uses a reverse lexicographic order as tie-breaker, but it reverses the outcome of the lexicographic comparison so that lexicographically larger monomials of the same degree are considered to be degrevlex smaller.*

2 Division and remainder in multi-variable rings

Compute the remainder on division of the given polynomial f by the set F. Use the lex order.

(1) Compute a remainder on division of the given polynomial f by the set F (by hand).

  (a) $f = x^7y^2 + x^3y^2 - y + 1, \quad F = \left(xy^2 - x, x - y^3\right)$.

  (b) Can you find a different remainder of this division?

(2) Compute a remainder on division:

  (a) $f = xy^2z^2 + xy - yz, \quad F = \left(x - y^2, y - z^3, z^2 - 1\right)$.

(b) Can you find a different remainder of this division?

(3) With the help Sage, repeat (1) and (2), but use the grevlex order instead.
Hint: *You may have a look on the examples presented in the exercise sheet to see how Sage handles these kinds of computations.*

This exercise is an adaption of Exercises 1-3 in Chapter 2, §3 of Cox, Little + O'Shea "Ideals, varieties and algorithms"

3 Buchberger's Criterion

Using Buchberger's Criterion, determine whether the following sets G are Gröbner bases for the ideal they generate. You may want to use Sage to compute the S-polynomials and remainders.

(a) $G = \{x^2 - y, x^3 - z\}$ for grlex order.

(b) $G = \{xy^2 - xz + y, xy - z^2, x - yz^4\}$ for lex order.

This exercise is an adaption of Exercises 10 in Chapter 2, §6 of Cox, Little + O'Shea "Ideals, varieties and algorithms"