Exercises: Week 6

Computation in Algebra and Arithmetic

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Sage provides standard constructions for polynomials. A brief introduction on how they are implemented can be found here: `https://doc.sagemath.org/html/en/tutorial/tour_polynomial.html`.

1 Gröbner Bases correspond to RREF

Let $A = (a_{ij})$ be an $n \times m$ matrix with entries in k and let $f_i = a_{i1}x_1 + \cdots + a_{im}x_m$ be the linear polynomials in $k[x_1, \ldots, x_m]$ determined by the rows of A. Then we get the ideal $I = \langle f_1, \ldots, f_n \rangle$. We will use lex order with $x_1 > \cdots > x_m$. Now let $B = (b_{ij})$ be the reduced row echelon matrix determined by A and let $g_1, \ldots, g_t$ be the linear polynomials coming from the nonzero rows of B (so that $t \leq n$). We want to prove that $g_1, \ldots, g_t$ form the reduced Gröbner basis of I.

(a) Show that $I = \langle g_1, \ldots, g_i \rangle$. **Hint:** Show that the result of applying a row operation to A gives a matrix whose rows generate the same ideal.

(b) Use Buchberger's Criterion to show that $g_1, \ldots, g_t$ form a Gröbner basis of I. **Hint:** If the leading 1 in the i th row of B is in the s th column, we can write $g_i = x_s + C$, where C is a linear polynomial involving none of the variables corresponding to leading l's. If $g_j = x_\ell + D$ is written similarly, then you need to divide $S(g_i, g_j) = x_\ell D - x_3 D$ by $g_1, \ldots, g_i$. Note that you will use only g_i and g_j in the division.

(c) Explain why $g_1, \ldots, g_t$ form the reduced Gröbner basis of I.

This exercise is an adaption of Exercises 10 in Chapter 2, §7 of Cox, Little + O'Shea "Ideals, varieties and algorithms"

2 Elimination theory

2.1 Warm-up

Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal.

(a) Prove that $I_l = I \cap k[x_{l+1}, \ldots, x_n]$ is an ideal of $k[x_{l+1}, \ldots, x_n]$.

(b) Prove that the ideal $I_{l+1} \subseteq k\left[x_{l+2}, \ldots, x_n\right]$ is the first elimination ideal of $I_l \subseteq k\left[x_{l+1}, \ldots, x_n\right]$.

This exercise is an adaption of Exercises 1 in Chapter 3, §1 of Cox, Little + O'Shea "Ideals, varieties and algorithms"

2.2 Images of algebraic sets

Consider the map $\bar{F}^2 \to \bar{F}^3$ given by $(x_1, x_2) \to (x_1 x_2, x_1^2, x_2^2)$. Compute the Zariski closure of the image of $\mathbf{V}((x_1 - x_2))$ and of $\mathbf{V}((x_1^3 - x_2 + 1))$.

3 Solving polynomial Equations

Find the points in $\mathbb{C}^3$ on the variety $\mathbf{V}\left(x^2 + y^2 + z^2 - 1, x^2 + y^2 + z^2 - 2x, 2x - 3y - z\right)$.

This exercise is an adaption of Exercises 3 in Chapter 2, §8 of Cox, Little + O'Shea "Ideals, varieties and algorithms"