Exercises: Week 7

Computation in Algebra and Arithmetic

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Sage provides standard constructions for polynomials. A brief introduction on how they are implemented can be found here: `https://doc.sagemath.org/html/en/tutorial/tour_polynomial.html`.

1 Hilbert Polynomials

1. Let k be a field. Show that a nonzero ideal $I \subset k[x_1, ..., x_n]$ has infinite dimension as a vector space over k.
2. In this exercise, we will show that if $I_1 \subseteq I_2$ are ideals in $R = k[x_1, \ldots, x_n]$, then

$$\deg(HP_{I_1}) \geq \deg(HP_{I_2}).$$

   For each monomial ideal I, we let

$$C(I) = \left\{\alpha \in \mathbb{Z}^n_{\geq 0} \mid x^\alpha \notin I\right\}.$$

   (a) Show that $I_1 \subseteq I_2$ implies $C(\langle \mathrm{LT}(I_1)\rangle) \supseteq C(\langle \mathrm{LT}(I_2)\rangle)$ in $\mathbb{Z}^n_{\geq 0}$.

   (b) Show that for $s \geq 0$, the Hilbert functions satisfy the inequality

$$HF_{I_1}(s) \geq HF_{I_2}(s).$$

   (c) From part (b), deduce the desired statement about the degrees of the Hilbert polynomials. Hint: Argue by contradiction and consider the values of the polynomials as $s \to \infty$.

3. Compute the (affine) Hilbert polynomials for each of the following ideals.

   a) $I_1 = \langle x^3y, xy^2\rangle \subseteq k[x, y]$.

   b) $I_2 = \langle x^3yz^5, xy^3z^2\rangle \subseteq k[x, y, z]$.

   Note: *There are some technical subtleties to be regarded when using sage to compute Hilbert polynomials as sage only knows how to compute projective Hilbert polynomials. See also section* $6.4\frac{1}{2}$ *in the script.* **In particular, the Hilbert Polynomial of I_1 is not given by 2, but by $2t + 3$.**

4. Find the index of regularity [that is, the smallest s_0 such that $HF_I(s) = HP_I(s)$ for all $s \geq s_0$] for each of the ideals in the last exercise.

5. Consider the following system of equations over $\bar{\mathbb{Q}}$

$$\begin{aligned} 3x^2 + 2yz - 2x\lambda &= 0 \\ 2xz - 2y\lambda &= 0 \\ 2xy - 2z - 2z\lambda &= 0 \\ x^2 + y^2 + z^2 - 1 &= 0. \end{aligned}$$

   First verify that it has only finitely many showing by proving that the corresponding Hilbert polynomial is constant. Then solve the system of equations using a Gröbner basis with respect to a suitable order.

Some of these exercises are fully or partly taken copied from Cox, Little + O'Shea "Ideals, varieties and algorithms"