Exercises: Week 8

Computation in Algebra and Arithmetic

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1 Data observation - prove your own formula!

The goal of this exercise is to guide you into discovering the structure of the 2-torsion parts of class groups for imaginary quadratic fields. If you manage to complete it, you will prove a formula you came up with yourself!

(a) Tabulate the sizes of the 2-torsion parts of class groups for imaginary quadratic fields of discriminant up to 100 with the help of Sage or PARI/GP.

(b) Try to guess a formula for the size of the 2-torsion parts of class groups for imaginary quadratic fields.

(c) Try out your formula for a few imaginary quadratic fields with higher discriminant.

(d) Prove the conjectured formula.

2 Computing class groups

Using the Minkowski bound, compute the class groups of the following imaginary quadratic fields. You can check your result using Sage or the LMFDB.

(a) $\mathbf{Q}(\sqrt{-5})$;

(b) $\mathbf{Q}(\sqrt{-21})$;

(c) $\mathbf{Q}(\sqrt{-41})$.

3 A fun application of the class number

Show that $x^3 - y^2 = 13$ has no integral solutions but $(17, \pm70)$.
Hint: Look at the class number of $\mathbf{Q}(\sqrt{-13})$.