Exercise Sheet 13.

Algebraic geometry

25.05.2022

Definition 0.1. 1. A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and a map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

2. A locally ringed space $(X, \mathcal{O}_X)$ is a ringed space where for each $x \in X$, $\mathcal{O}_{X,x}$ is a local ring. A morphism of locally ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.

Definition 0.2. A scheme $(X, \mathcal{O}_X)$ is a locally ringed space that is locally isomorphic to an affine scheme, i.e., for each $x \in X$, there exists an open subset $U \subset X$ such that $U \cong \mathrm{Spec} R$ for some ring R.

Q1 Give an example of a morphism of ringed space that is not a morphism of schemes.

Q2 Show that any closed irreducible subset of a scheme has a unique generic point.

Q3 Show that a scheme X is integral (i.e., irreducible and $\mathcal{O}_{X,x}$ is reduced for all $x \in X$) if and only if $\mathcal{O}_X(U)$ is integral domain for any affine open subset $U \subset X$.

Q4 Let p be a prime. For a scheme X over $\mathbb{F}_p$, define $F : X \to X$ be the identity map on the underlying topological space and $F^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ is defined by $F^\sharp(g) = g^p$. Show that it is a map of schemes.