Exercise Sheet 2.

Algebraic geometry

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Let k be an algebraically closed field. Some exercises below are from [Ellingsrud-Ottem].

Q1 (2.3) Show that any nonempty Hausdorff space has dimension 0.

Q2 (2.5) Assume that $Y = Y_1 \cup \cdots \cup Y_r$ is the decomposition of the Noetherian space Y into irreducible components. Show that $\dim Y = \max \dim Y_i$ for Y_i irreducible components.

Q3 (2.10) Let $\psi : \mathbb{A}^2_k \to \mathbb{A}^2_k$ be the map $\psi(x, y) = (x, xy)$. Determine the ideals $\psi^* \mathfrak{m}_{(a,b)}$ and the fibres $\psi^{-1}(a, b)$ for all points $(a, b) \in \mathbb{A}^2_k$.

The following two questions are about presheaves and sheaves.

Definition 0.1. Let X be a topological space. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves (of abelian groups) on X, a morphism $\phi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of abelain groups $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open subset $U \subset X$ such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \end{array}$$

is commutative. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, a morphism $\mathcal{F} \to \mathcal{G}$ is a morphism of presheaves.

Let $\mathrm{PreShv}(X)$ be the category of presehaves on X and let $\mathrm{Shv}(X)$ be the category of sheaves on X.

Q4 Let X be a topological space. Show that the inclusion functor $\mathrm{Shv}(X) \to \mathrm{PreShv}(X)$ has a left adjoint.

Q5 (3.3) Consider the real line $\mathbb{R}$ with the Euclidean topology.

a) Let $\mathcal{B}$ be the presheaf of bounded continuous real valued functions on $\mathbb{R}$, show that $\mathcal{B}$ is not a sheaf.

b) What is the sheafification of $\mathcal{B}$?

Hints/Suggestions

Q1 We first prove that any point $x \in X$ is closed. Consider an open subset

$$U = \bigcup_{y \neq x} U_y$$

where U_y is an open neighborhood of y which does not contain x. Clearly $X - U = \{x\}$ and hence x is closed. Next let $X_0 \subset X$ be a closed subset with $|X_0| \geq 2$. Then it is easy to see that X_0 is reducible. Therefore maximal length of a chain of closed irreducible subset of X is one.

Q2 The statement is clear from the following lemma.

Lemma 0.2. (1) If $Z \subset X$ is irreducible, then $\overline{Z}$ is an irreducible subset of X.
(2) Any irreducible subset is contained in an irreducible component.

Q3 See lecture note (chapter 2).

Q4 We prove this for presheaves of sets on X. Other cases follow easily from this case. Sheaves on X satisfy extra two conditions:

(1) If $U = \cup_{i \in I} U_i$ then
$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$$
is injective.

(2) If $U = \cup_{i \in I} U_i$ then
$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$
is exact.

Let $\mathrm{PreShv}^*(X)$ is the category of presheaves on X satisfying (1).
[Step 1] The natural functor $\mathrm{PreShv}^*(X) \to \mathrm{PreShv}(X)$ has a left adjoint functor.
Let $\mathcal{F}$ be a presheaf on X. For each open subset $U \subset X$ consider

$$\mathcal{F}^s(U) = \mathcal{F}(U)/\sim$$

where $a \sim b$ if and only if there exists a covering $U = \cup_{i \in I} U_i$ such that a, b have the same image under

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i).$$

It is straightforward to check it induces a left adjoint functor $(-)^s : \mathrm{PreShv}(X) \to \mathrm{PreShv}^*(X)$.
[Step 2] The natural functor $\mathrm{Shv}(X) \to \mathrm{PreShv}^*(X)$ has a left adjoint functor.
Let $\mathcal{F}$ be a presheaf on X satisfying (1). For each open subset $U \subset X$ we consider

$$\mathcal{F}^a(U) = \{(U = \cup_{i \in I} U_i, \{a_i\}_{i \in I}) : \{a_i\} \in \mathrm{Eq}(\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j))\}/\sim$$ [1]

where

$$(U = \cup_{i \in I} U_i, \{a_i\}) \sim (U = \cup_{j \in I} V_j, \{b_j\})$$

iff a_i, b_j have the same image in $\mathcal{F}(U_i \cap V_j)$ under $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap V_j)$, $\mathcal{F}(V_j) \to \mathcal{F}(U_i \cap V_j)$ for all $i \in I, j \in J$. It is straightforward to check that it induces a left adjoint functor $(-)^a : \mathrm{PreShv}^*(X) \to \mathrm{Shv}(X)$.

Q5 This is a simple application of Q4.
a) Locally bounded continuous function may not be bounded globally.
b) Let $\mathcal{G}$ be the sheaf of continuous functions on X. Then the natural morphism $f : \mathcal{B} \to \mathcal{G}$ factors through a unique morphism $f^a : \mathcal{B}^a \to \mathcal{G}$. We use the following lemma

[1] Let $f, g : X \to Y$ be maps between sets. Then $\mathrm{Eq}(f, g : X \rightrightarrows Y) := \{x \in X : f(x) = g(x)\}$

Lemma 0.3. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then f is an isomorphism if and only if it is an isomorphism at each stalk $x \in X$.

It is clear that f^a induces an isomorphism at each stalk $x \in X$. Hence f^a is an isomorphism of sheaves.