

Q1

Thm (Luroth)  $L \subseteq k(t)$  s.t.  $\text{tr. deg}_k(L) = 1 \Rightarrow \exists u \in L$  s.t.  $L \cong k(u)$

Fact Let  $C$ : regular projective curve/ $k$ . If  $k(C) \cong k(t)$  then  $C \cong \mathbb{P}_k^1$ .

Suppose  $C$ : regular, proj. curve/ $k$  s.t.

$f: \mathbb{P}_k^1 \dashrightarrow C$  dominant, rational map.

$\Rightarrow f^*: k(C) \hookrightarrow k(\mathbb{P}_k^1) \cong k(t)$

$\dim C = 1 \Rightarrow \text{tr. deg}_k(k(C)) = 1 \Rightarrow k(C) \cong k(u) \exists u \in k(C)$  s.t.  $k(C) \cong k(u)$   
 $\Rightarrow C \cong \mathbb{P}_k^1$

Remark

It does not mean that  $f$  is an isomorphism. e.g.

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^1 \quad [x_0 : x_1] \longmapsto [x_0^d : x_1^d] \quad u = t^d.$$

char(k) ≠ 2

Q2 C = Z(y^2 - x^3 - x^2)

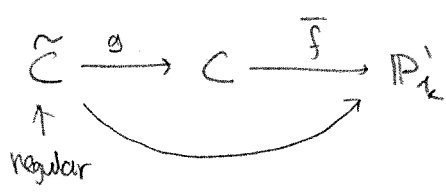
f: C \ {0,0} -> P^1\_k (x,y) -> [x:y]

Idea y = +/- x\*sqrt(x+1) so as (x,y) -> 0, f(0) = [1:1] or [1:-1]

Thm C: regular curve p in C, f: C \ {p} -> P^N given. Then f uniquely extends to

f\_bar: C -> P^N

"Formal approach." Suppose f: C \ {0,0} -> P^1\_k extends to C. Take a normalization



g: Isom on the smooth locus of C (ie C \ {0,0})

Show that the preimage of (0,0) under g maps to two different points [1:1] & [1:-1] under f\_bar o g which leads to a contradiction.

Q3 Lemma  $X$ : separated. Then any ~~an~~ open or closed subvariety of  $X$  is separated

Let  $V \xrightarrow{i} X$  open or closed

$$\begin{array}{ccc} V & \longrightarrow & V \times V \\ \downarrow i & & \downarrow i \times i \\ X & \longrightarrow & X \times X \end{array} \quad \Delta_V = (i \times i)^{-1}(\Delta_X) : \text{closed}$$

Since  $A_k^n$  is separated, any affine variety is separated

To show projective varieties are separated, it is enough to show that  $\mathbb{P}_k^n$  is separated.

Nice coordinate on  $\mathbb{P}_k^n \times \mathbb{P}_k^m$ : Segre embedding

Def  $f: \mathbb{P}_k^n \times \mathbb{P}_k^m \longrightarrow \mathbb{P}_k^N \quad N = (n+1)(m+1) - 1$

$$[x_0: \dots: x_n] [y_0: \dots: y_m] \longmapsto [z_{ij}]_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}}$$

$\Rightarrow \text{Image } f(\mathbb{P}_k^n \times \mathbb{P}_k^m) = Z(z_{ij}z_{kl} - z_{il}z_{kj} : 0 \leq i, k \leq n, 0 \leq j, l \leq m)$

( $\subseteq$ )  $\checkmark$

( $\supseteq$ ) at least one of them should be nonzero (wlog  $z_{00} = 1$ )

$\Rightarrow z_{ij} = z_{i0}z_{0j}$ . Let  $x_i := z_{i0}$ ,  $y_j := z_{0j}$ . Then  $\exists z \in \mathbb{P}_k^n \times \mathbb{P}_k^m$  maps to  $f(z) = Z(\dots)$

Similarly one can show that  $\mathbb{P}_k^n \times \mathbb{P}_k^m \longrightarrow f(\mathbb{P}_k^n \times \mathbb{P}_k^m)$  is bijective &  $f^{-1}$ : poly.  $\square$

Claim  $\mathbb{P}_k^n \longrightarrow \mathbb{P}_k^n \times \mathbb{P}_k^n$  closed.

View everything inside  $\mathbb{P}_k^N$  via Segre embedding then

$$\Delta_{\mathbb{P}^n} = Z(z_{ij} - z_{ji}) \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^n, \quad z_{ij} = x_i y_j \quad \square$$

Cor Any quasi-projective varieties are separated.

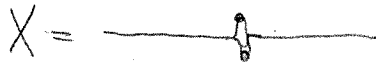
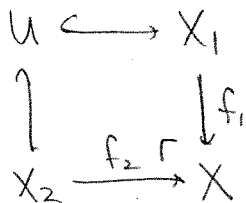


Separatedness is a global property.

Remark Suppose  $X = U_1 \cup U_2$ ,  $U_i = \text{separated open in } X$ . Then  $X$  may not be separated.

Example (Affine line with a double origin)

$$X_1 = X_2 = \mathbb{A}^1, U = \mathbb{A}^1 - 0$$



Claim  $X$  is not separated

If it were...

$$\begin{array}{ccc}
 & & X \\
 & & \downarrow \Delta \\
 X_1 = X_2 & \xrightarrow{f_1 \times f_2} & X \times X
 \end{array}$$

$$(f_1 \times f_2)^{-1}(\Delta_X) \cong U, \text{ not closed}$$

