Q1

Thm (Lüroth) $L \subseteq k(t)$ s.t $tr.deg_k(L) = 1 \Rightarrow \exists u \in L$ s.t $L \cong k(u)$

Fact Let C : regular projective curve / k. If $k(C) \cong k(t)$ then $C \cong \mathbb{P}^1_k$.

Suppose C : regular, proj. curve / k s.t

$$f : \mathbb{P}^1_k \dashrightarrow C \qquad \text{dominant, rational map.}$$

$$\Rightarrow f^* : k(C) \hookrightarrow k(\mathbb{P}^1_k) \cong k(t)$$

$\dim C = 1 \Rightarrow tr\,deg_k(k(C)) = 1 \Rightarrow k(C) \cong k(u) \quad \exists\, u \in k(C)$ s.t $k(C) \cong k(u)$

$\Rightarrow C \cong \mathbb{P}^1$.

Remark

It does not mean that f is an isomorphism. e.g

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^1 \qquad [x_0 : x_1] \longmapsto [x_0^d : x_1^d] \qquad u = t^d.$$

char$(k) \neq 2$

Q2 $C = Z(y^2 - x^3 - x^2)$

- $f : C \setminus \{(0,0)\} \longrightarrow \mathbb{P}^1_k \quad (x,y) \longmapsto [x:y]$

Idea $y = \pm x\sqrt{x+1}$ so as $(x,y) \to 0$. $f(0) = [1:+1]$ or $[1:-1]$.

Thm C : regular curve $P \in C$. $f : C \setminus P \longrightarrow \mathbb{P}^N$ given. Then f uniquely extends to

$$\bar{f} : C \longrightarrow \mathbb{P}^N$$

"Formal approach." Suppose $f : C \setminus \{0,0\} \longrightarrow \mathbb{P}^1_k$ extends to C. Take a normalization

$$\begin{tikzcd} \tilde{C} \arrow[r, "g"] \arrow[rr, bend right] & C \arrow[r, "\bar{f}"] & \mathbb{P}^1_k \end{tikzcd}$$

$\uparrow$
regular

g : Isom on the smooth locus of C (ie $C \setminus \{0,0\}$)

Show that the preimage of $(0,0)$ under g maps to two different points $[1:1]$ & $[1:-1]$ under $\bar{f} \circ g$ which leads to a contradiction.

Q3 <u>Lemma</u> X : separated. Then any ~~on~~ open or closed subvariety of X is separated

Let $V \overset{i}{\hookrightarrow} X$ open or closed

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \times V \arrow[d, "i \times i"] \\ X \arrow[r] & X \times X \end{tikzcd} \qquad \Delta_V = (i \times i)^{-1}(\Delta_X) \ : \text{closed}$$

Since $\mathbb{A}^n_k$ is separated, any affine variety is separated

To show projective varieties are separated, it is enough to show that $\mathbb{P}^n_k$ is separated.

Nice coordinate on $\mathbb{P}^n_k \times \mathbb{P}^m_k$: <u>Segre embedding</u>

<u>Def</u> $\quad f : \mathbb{P}^n_k \times \mathbb{P}^m_k \longrightarrow \mathbb{P}^N_k \qquad N = (n+1)(m+1) - 1$

$$[x_0 : \cdots x_n]\ [y_0 \cdots y_m] \longmapsto [z_{ij}]_{\substack{0 \le i \le n \\ 0 \le j \le m}}$$

$\Rightarrow$ Image $f(\mathbb{P}^n_k \times \mathbb{P}^m_k) = Z(z_{ij} z_{kl} - z_{il} z_{kj} : 0 \le i, k \le n,\ 0 \le j, l \le m)$

($\subseteq$) v

($\supseteq$) at least one of them should be nonzero wolog $z_{00} = 1$

$\Rightarrow z_{ij} = z_{i0} z_{0j}$. Let $x_i := z_{i0}$, $y_j = z_{0j}$. Then $\exists\, z \in \mathbb{P}^n_k \times \mathbb{P}^m_k$ maps to ~~f(…)~~ $Z(\cdots)$.

Similarly one can show that $\mathbb{P}^n_k \times \mathbb{P}^m_k \longrightarrow f(\mathbb{P}^n_k \times \mathbb{P}^m_k)$ is bijective & f^{-1} : poly. ▨

<u>Claim</u> $\mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k \times \mathbb{P}^n_k$ closed.

View everything inside $\mathbb{P}^N_k$ via Segre embedding the

$$\Delta_{\mathbb{P}^n} = Z(z_{ij} - z_{ji}) \subseteq \mathbb{P}^n_k \times \mathbb{P}^n_k, \quad z_{ij} = x_i y_j \ ▨$$

<u>Cor</u> Any quasi-proj varieties are separated.

Separatedness is a global property

<u>Remark</u> Suppose $X = U_1 \cup U_2$, U_i = separated open in X. Then X may not be separated.

<u>Example</u> (Affine line with a double origin)

$X_1 = X_2 = \mathbb{A}^1$. $U = \mathbb{A}^1 - 0$

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, hook] & X_1 \arrow[d, "f_1"] \\ X_2 \arrow[r, "f_2"] \arrow[ur, phantom, "\ulcorner"] & X \end{tikzcd}$$

$X =$ 

<u>Claim</u> X is not separated

If it were...

$$\begin{tikzcd} & X \arrow[d, "\Delta"] \\ X_1 = X_2 \arrow[r, "f_1 \times f_2"] & X \times X \end{tikzcd}$$

$(f_1 \times f_2)^{-1}(\Delta_X) \cong U$. not closed

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