k = alg closed

Q1 $\phi : \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{d} \longrightarrow \mathbb{P}^d$

$([x_1 : y_1], \ldots, [x_d : y_d]) \longmapsto \text{Coeff} \prod_{i=1}^{d} (x_i - y_i s)$ s : formal variable

We know $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ & $\mathbb{P}^d$: projective $\Rightarrow$ enough to show that ϕ is quasi-finite

Wlog $y_i \neq 0 \ \forall i = 1, \ldots, d$ so may assume $y_i = 1$

$\Rightarrow$ Coeff $\prod_{i=1}^{d} (x_i - s)$ is fixed by $a_0 \cdots a_{d-1} \in k$

$$\prod_{i=1}^{d} (x_i - s) = s^d + a_{d-1} s^{d-1} + \cdots + a_1 s + a_0$$

By fundamental thm of alg the RHS has finitely many solution. In fact possible number of tuples $= |S_d| = d!$ (for general choice of a_i)

Q2) $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ $z \longmapsto [f(z) : g(z)]$. no common zero, What is $\deg(\phi)$

$\deg(\phi) :=$ deg of field extension.

May assume $\deg f > \deg g$. & f, g : monic polynomial

<u>Claim</u> $\deg(\phi) = \deg(f)$

Pf) Let $h = \frac{f}{g} \in k(z)$. $k(h) \subset k(z)$. $\deg(\phi) = [k(z) : k(h)]$

Enough to show : The minimal polynomial of z over $k(h)$ is given by

$$m_z(T) = f(T) - hg(T).$$

- $m_z(z) = f(z) - hg(z) = 0$. $k[h][T]$
  $m_z(T)$ is irred poly in $k[h, T]$ because it is linear in h

By Gauss lemma, $m_z(T)$ is irred in $\mathrm{Frac}(k[h])[T] = k(h)[T]$

$\Rightarrow$ $[k(z) : k(h)] = \deg(f)$ ▨

Q3.1 $M_0 = 0 \subset M_1 \subset M_2 \subset \cdots \subset M$ s.t. M_i/M_{i-1} : free B-module

$\Rightarrow M$: free

Pf) Inductively all M_n is free B-module

$$0 \to M_{n-1} \to M_n \to M_n/M_{n-1} \to 0 \quad \text{split } (\because M_n/M_{n-1} = \text{free})$$

$$\phi_n$$

$\Rightarrow \bigoplus_{n=1}^{\infty} M_n/M_{n-1} \xrightarrow{\cong} M$ (Injective ✓, surjective : $\exists\, l$ st $x \in M_l$)

$B \to A$ satisfies (*) if $\forall$ f.g A-mod M, $\exists\, 0 \neq f \in B$ s.t M_f : free B_f-mod

3.2 B : noetherian integral domain

Let M : f.g B-mod. Choose a filtration:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M \quad \text{s.t}$$

$M_l/M_{l-1} \cong B/I_l$ for some ideal $I_l \subset B$.

(e.g. $M = B\langle x_1 \cdots x_l \rangle$. $M_1 := Bx_1 = B/I_1$, $I_1 = \{b \in B : bx_1 = 0\}$

B : domain $\Rightarrow \prod_{I_l \neq 0} I_l$ is non-trivial

$\Rightarrow \exists_{0 \neq} f \in \prod I_l$

$$(M_l/M_{l-1})_f = (B/I_l)_f = \begin{cases} B_f \\ 0 \end{cases} \quad \left(\because \frac{b+I_l}{f^m} = \frac{fb+I_l}{f^{m+1}} = \frac{I_l}{f^{m+1}} = 0\right)$$

Since $(\ )_f$ is exact functor,

$$0 \subset (M_1)_f \subset \cdots \subset (M_n)_f$$

By Q1. $M_f \cong \bigoplus (M_l/M_{l-1})_f$ ∎

Q2.3 B : noetherian domain $B \to A$: f.g. alg satisfies $(*)$

$\Rightarrow A[T]$ satisfies $(*)$

Pf) M f.g $A[T]$-module generated by $x_1 \cdots x_n$. Do induction on n.

Take

$$\left.\begin{array}{l} M_0 = 0 \\ M_1 = A\langle x_1 \cdots x_n \rangle \\ \vdots \\ M_n = M_{n-1} + TM_{n-1} \end{array}\right\} \quad 0 = M_0 \subset M_2 \subset \cdots \subset M \text{ : filtration as } A\text{-modules}$$

Then $M_{n+1}/M_n = (M_n + T \cdot M_n)/M_n$

$$\vdots$$

$$= (M_1 + TM_1 + \cdots + T^n M_1)/(M_1 + \cdots + T^{n-1} M_1)$$

$$= M_1/N_n, \quad N_n = \{x \in M_1 : T^n x \in M_n\} \subseteq M_1 \text{ ; A-submodule}$$

$N_1 \subset N_2 \subset \cdots \subset M_1$ stabilizes bc M_1 : noetherian A-module.

M_1/N_n : A-module satisfies $(*)$

$$0 \subset O_{n.0} \subset \cdots \subset O_{n.e_n} \subset M_1/N_n$$

s.t. $O_{n.i}/O_{n.i-1} \cong B/I_{n.i}$ prime $\quad \exists f \in \prod I_{n.i}$

$\Rightarrow (M_1/N_n)_f$: free B_f-module

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