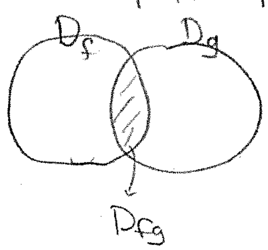
Recall (Affine scheme)

$$\mathrm{Spec} : (\text{Comm ring})^{\circ} \longrightarrow (\text{Locally ringed space})$$
$$R \longmapsto \mathrm{Spec}\, R$$

- Set : $\mathfrak{p} \subset R$ prime ideal
- Topology : $I \subset R$ ideal. closed subsets $V(I) = \{ \mathfrak{p} \subset R \text{ prime} : I \subset \mathfrak{p} \}$

  Basic open subsets $D_f = \{ \mathfrak{p} \subset R \text{ prime} : f \notin \mathfrak{p} \}$
- Structure sheaf $\mathcal{O}_{\mathrm{Spec}\, R}(D_f) = R_f$.



$$\begin{array}{ccc} \mathcal{O}_{\mathrm{Spec} R}(D_f) & \longrightarrow & \mathrm{Spec} R(D_{fg}) \\ \wr\wr & & \wr\wr \\ R_f & \longrightarrow & R_{fg} \end{array}$$

Eg $\mathcal{O}_{X,\mathfrak{p}} \cong R_{\mathfrak{p}}$.

Morphism : $R \xrightarrow{\varphi} S \rightsquigarrow {}^a\varphi : \mathrm{Spec}\, S \longrightarrow \mathrm{Spec}\, R$. map of locally ringed space.

Idea Identify "Space" with its "ring of functions on the space".

Now we can distinguish $\mathbb{k}$ and $\mathbb{k}[t]/(t^n)$; · vs ▬

Q1 $\operatorname{Spec} \mathbb{Z}[x] \longrightarrow \operatorname{Spec} \mathbb{Z}$

⚠ Geometry over Spec $\mathbb{Z}$ could be very difficult bc you see all the primes

Recall $\operatorname{Spec} \mathbb{Z}$: 2, 3, 5, 7, 11, …, (0)

$\mathbb{Z}$: initial object in (Rings) $\rightsquigarrow$ $\operatorname{Spec} R \longrightarrow \operatorname{Spec} \mathbb{Z}$ exists uniquely.

$\mathbb{Z} \overset{\varphi}{\hookrightarrow} \mathbb{Z}[x]$. canonical map. $\mathfrak{p} \subset \mathbb{Z}[x]$: prime ideal

Case 1) $\mathfrak{p} \cap \mathbb{Z} = (0)$

- $\mathfrak{p} = (0)$ Good
- $\mathfrak{p} \neq (0)$; $\mathfrak{p} \cap (\mathbb{Z} - \{0\}) = \phi$. Let $S = \mathbb{Z} - 0$. $S^{-1}\mathfrak{p} \subset \mathbb{Q}[x]$ prime ideal.

  $\mathbb{Q}$: field $\Rightarrow$ $S^{-1}\mathfrak{p} = (f)$ for some monic irred poly $f \in \mathbb{Q}[x]$

  Clear out denominators, take $f \in \mathbb{Z}[x]$ s.t $\gcd(\text{coeff } f) = 1$.

  By Gauss lemma, f is irred poly in $\mathbb{Z}[x]$, and $\mathfrak{p} = (f)$

Case 2) $\mathfrak{p} \cap \mathbb{Z} = (p)$ $p \geq 2$ prime

Consider $\mathbb{Z}[x] \overset{\pi_p}{\twoheadrightarrow} \mathbb{Z}/(p)\mathbb{Z}[x]$ $\pi_p(\mathfrak{p})$ is prime ideal ($\because \pi_p$ is onto)

- $\pi_p(\mathfrak{p}) = 0$; $\mathfrak{p} = (p)$
- $\pi_p(\mathfrak{p}) = (f)$, $f \in \mathbb{F}_p[x]$ monic irred polynomial.

  Take $\tilde{f} \in \mathbb{Z}[x]$ s.t $\pi_p(\tilde{f}) = f$, $\tilde{f}$ = monic, irreducible

  Claim $\mathfrak{p} = (p, \tilde{f})$

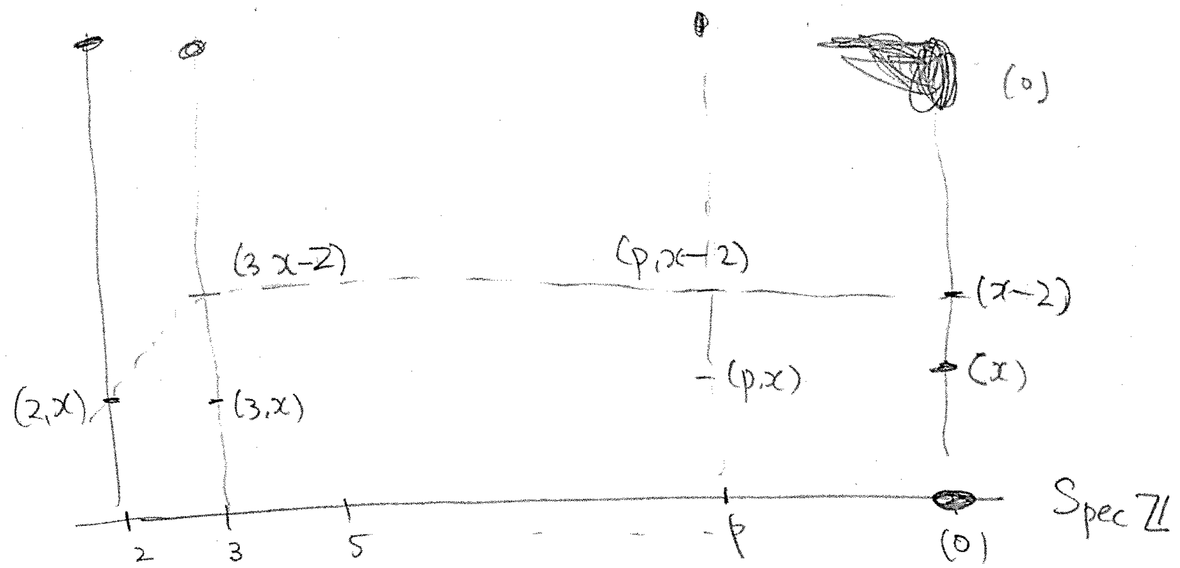
  $(\supseteq)$: Clear

  $(\subseteq)$ Let $g \in \mathfrak{p}$ $\Rightarrow$ $\pi_p(g) = f \cdot h$ $\exists h \in \mathbb{F}_p[x]$ Take any lift $\tilde{h}$ of h

  $\pi_p(g - \tilde{f}\tilde{h}) = \pi_p(g) - fh = 0$

  $\Rightarrow (g - \tilde{f}\tilde{h}) \in (p)$ $g = \tilde{f}\tilde{h} + p\varepsilon$. $\varepsilon \in \mathbb{Z}[x]$.

  $\Rightarrow g \in (p, \tilde{f})$



<u>Q2</u> $\operatorname{Spec} R \sqcup \operatorname{Spec} S$ homeo to $\operatorname{Spec}(R \times S)$

Consider

$$R \xrightarrow{q_1} R \times S \xrightarrow{p_1} R$$
$$r \longmapsto (r, 1)$$
$$S \xrightarrow{q_2} R \times S \xrightarrow{p_2} S$$

$\rightsquigarrow$

$$\begin{tikzcd} \operatorname{Spec} R \sqcup \operatorname{Spec} S \arrow[r, "p^*"] \arrow[dr, "id"'] & \operatorname{Spec}(R \times S) \arrow[d, "q^*"] \arrow[dl, phantom, "?"] \\ & \operatorname{Spec} R \sqcup \operatorname{Spec} S \end{tikzcd}$$

Enough to show the following

<u>Claim</u> p^* (or q^*) is a bijection

Pf) If $I = \mathfrak{p} \times S$ or $R \times \mathfrak{q}$, then $I \subset R \times S$ is a prime ideal.

Suppose $I \subset R \times S$ is a prime ideal. Then $(1,0)$ or $(0,1) \notin I$, so wlog $(1,0) \notin I$

$(1,0) \cdot (0,1) = 0 \in I \Rightarrow (0,1) \in I$, so $0 \times S \subset I$

p_1 : surjective $\Rightarrow p_1(I) \subset R$: prime ideal

$I \subset p_1^{-1} p_1(I) = p_1(I) \times S$ ✓

$p_1(I) \times 0 \subset I$ & $0 \times S \subset I \Rightarrow p_1(I) \times S \subset I$ ☑

Q3 X : affine scheme / $k = \bar{k}$

Lemma (R, m) local ring $X = \operatorname{Spec} A$. Then $\exists$ bijection

$$\operatorname{Hom}_{Sch}(\operatorname{Spec} R, \operatorname{Spec} A) \longleftrightarrow \left\{ (x, \varphi) : x \in \underset{\operatorname{Spec} A}{X}, \ \varphi : \mathcal{O}_{X,x} \to R \text{ local hom} \right\}$$

ie $\varphi^{-1}(m) = m_x$.

sketch • $f : \operatorname{Spec} R \to \operatorname{Spec} A$, $m \mapsto x$ $\rightsquigarrow$ $f^{\#} : \mathcal{O}_{\operatorname{Spec} A, x} \to \mathcal{O}_{\operatorname{Spec} R, m} \simeq R$.

• $A \xrightarrow{\varphi} R$. Then $\exists!$ prime ideal $p \subset A$ s.t
$$\begin{tikzcd} A \arrow[r] \arrow[dr] & R \\ & A_p \arrow[u, "\text{local hom}"'] \end{tikzcd}$$
ie $p = \varphi^{-1}(m)$

$$\operatorname{Hom}_{Sch/k}(\operatorname{Spec} k[t]/t^2, X) \Longleftrightarrow \{ (x, \varphi) : x \in X, \ \varphi : \mathcal{O}_{X,x} \to k[t] \}$$

$\varphi(m_x) \subset (t)$

$\varphi(m_x^2) = (0)$

Since $\varphi(m_x^2) = 0$, enough to assign a map

$\Rightarrow$ $\mathcal{O}_{X,x}/m_x^2 \longrightarrow k[t]$

Consider

$$0 \to m_x/m_x^2 \to \mathcal{O}_{X,x}/m_x^2 \to \underset{\underset{k}{\cong}}{\mathcal{O}_{X,x}/m_x} \to 0$$

$\mathcal{O}_{X,x}/m_x^2 \cong m_x/m_x^2 \oplus k$

Then $\mathcal{O}_{X,x}/m_x^2 \to k[t] \Longleftrightarrow$ k-linear map $m_x/m_x^2 \longrightarrow k(t)$