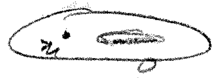
<u>Q1</u> $(R, m, k)$ : DVR. $\operatorname{Spec} R = \{m, 0\}$  e.g. $k[[t]]$

$K = \operatorname{Frac}(R)$ $\qquad R_m = R \quad R_0 \cong K$

Take: $f : \operatorname{Spec} K \longrightarrow \operatorname{Spec} R \qquad f(0) = \{m\}$.

$\qquad f^\# : R \longrightarrow K \qquad (f^\#)^{-1}(0) = 0$ not a maximal ideal in $R$

$\Rightarrow (f, f^\#)$ is a morphism of rined spaces but not <u>locally ringed spaces</u>

Q2 X : scheme $\emptyset \neq Z \subset X$ closed and irred $\Rightarrow \exists!\ \eta_Z \in Z$ generic point.

Pf) Cover up Z by affine open subsets $U_i \subset X$ $i \in I$ ie $Z \subset \bigcup_{i \in I} U_i$

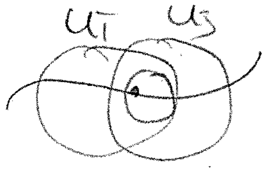
May assume $Z \cap U_i \neq \emptyset$ $\forall i \in I$. $\exists!$ generic point $\eta_i \in Z \cap U_i$

ie $\overline{\eta_i} \cap U_i = Z \cap U_i$



Claim $\eta_i = \eta_j$ $\forall i, j \in I$

Pf) First show that they are in $U_i \cap U_j$ $(i \neq j)$





If $\eta_i \notin U_j \Rightarrow \eta_i \in U_i \setminus U_j \subsetneq U_i$

$\Rightarrow \overset{Z_i}{\overline{\eta_i}} \cap U_i \subsetneq U_i \setminus U_j$

$\Rightarrow \overline{\eta_i} \cap U_i \cap U_j = \emptyset$ ↯ $Z$ = irred $\therefore \eta_i \in U_i \cap U_j$.

Second we show $\eta_i = \eta_j$. Let $\eta_i \in U_{ij} \subset U_i \cap U_j$. $U_{ij}$ : affine open

$\overline{\eta_i} \cap U_{ij} = (Z \cap U_i \cap U_j) \cap U_{ij} = \overline{\eta_j} \cap U_{ij} \Rightarrow \eta_i = \eta_j$ (uniqueness of generic pt) ▨

$\eta := \eta_i$ $\forall i \in I$

$\therefore Z = \overline{\eta}$

Uniqueness If $Z = \overline{\eta} = \overline{\eta'}$ Take $\overline{\eta} \cap U_i = \overline{\eta'} \cap U_i \Rightarrow \eta = \eta'$ by uniqueness. ▨

$\underline{\text{Q3}}$ $X$ : Integral scheme (ie $\overset{\neq\emptyset}{\text{Irred}}$, $\mathcal{O}_{X,x}$ : reduced $\forall x \in X$) iff $\forall \emptyset \neq \operatorname{Spec} R \cong U \underset{\text{open}}{\subseteq} X$

$R$ is domain.

Pf) ($\Leftarrow$) $R$ : domain, $\mathfrak{p} \subset R$ prime ideal $\Rightarrow R_{\mathfrak{p}}$ : domain ($\because R_{\mathfrak{p}} \hookrightarrow \operatorname{Frac}(R)$).

Suppose $X$ is <u>not</u> Irred. Then $\exists U_1, U_2 \hookrightarrow$ open $X$, disjoint, nonempty affine.

Thus

$$\mathcal{O}_X(U_1 \sqcup U_2) = \mathcal{O}(U_1) \times \mathcal{O}(U_2) \leftarrow \text{not domain} \ ↯$$

$\therefore X$ : integral

($\Rightarrow$) First show that $\mathcal{O}_X(U)$ is reduced $\forall U \subseteq X$. affine

$\underline{\text{Lemma}}$ $R$ : reduced $\Leftrightarrow \forall \mathfrak{p} \subset A$ prime $R_{\mathfrak{p}}$ : reduced (prime $\rightsquigarrow$ maximal ideal is also fine)

($\Rightarrow$) Easy

($\Leftarrow$) If not. $\exists\, 0 \neq x \in R$ s.t $x^m = 0$, $m \geq 2$. $\operatorname{Ann}(x) \subsetneq R$ : proper ideal.

Zorn's lemma, $\exists$ maximal ideal $\mathfrak{m}$, $\operatorname{Ann}(x) \subseteq \mathfrak{m}$. $R \longrightarrow R_{\mathfrak{m}}$, $x \mapsto 0$

$\Rightarrow \exists\, \alpha \in R \setminus \mathfrak{m}$ s.t $\alpha x = 0$ So $\alpha \in \operatorname{Ann}(x)$. ↯ ▨

Suppose $\exists\, \emptyset \neq U = \operatorname{Spec} R \subset X$ open s.t $R$ is not integral. Then,

$\exists\, f, g \neq 0 \in R$ s.t $fg = 0$

$\Rightarrow U = V(f) \cup V(g)$. $\quad V(f) = \{x \in U : f(x) = 0\} \hookrightarrow$ closed $U$

$X$ : Irreducible $\Rightarrow U$ is Irreducible

$\Rightarrow$ either $U = V(f)$ or $V(g)$ So let $U = V(f)$. $D(f) = \emptyset$

$R_f = 0 \Leftrightarrow f \in R$ nilpotent ↯ ▨

Q4 $X/\mathbb{F}_p$ . $p \geq 2$ . $F: X \longrightarrow X$ . $F^{\#}(g) = g^p$.

We show the following lemma

Lemma $R$ : local ring , $p=0$ in $R$ Then $R \xrightarrow{\varphi} R$ $\varphi(x) = x^p$ is a local homomorphism ie $\varphi^{-1}(m_R) = m_R$

Pf) $R \setminus m_R = R^\times$ so enough to show $\varphi^{-1}(R \setminus m_R) = R \setminus m_R$

- $\alpha \in R^\times \Rightarrow \alpha^p \in R^\times$ ✓
- If $\alpha^p \in R^\times \Rightarrow \alpha \in R^\times$ ($\because \alpha^p u = 1 \Rightarrow \alpha(\alpha^{p-1}u) = 1$ )