

①

Q1 X : topological space

Def X is reducible if $X = X_1 \cup X_2$, $X_1, X_2 \subset X$ closed and proper ($\neq \emptyset$ or X).
 X is irreducible if it is not reducible.

$\dim X := \sup \{ X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_r \}$. $X_i \neq \emptyset$, closed irred subsets of X .

Claim X : nonempty Hausdorff space. Then $\dim X = 0$

Proof ① Let $X_0 \subset X$ any closed subset with $|X_0| \geq 2$

$x_1 \neq x_2 \in X_0$, nbh $x_1 \in U_1, x_2 \in U_2, U_1 \cap U_2 = \emptyset$.

$\Rightarrow X_0 = U_1^c \cup U_2^c$. $U_1^c, U_2^c \subset X_0$ closed. $\therefore X_0$ is ~~not~~ reducible

~~\Rightarrow possible chain $\{x\}$ (any singleton is irred)~~

② Any point $x \in X$ is closed.

$$U := \bigcup_{\substack{y \in X \\ y \neq x}} U_y = X - \{x\} \text{ is open}$$

$U_y \cap \{x\} = \emptyset$

\Rightarrow Possible chain $\{x\}$. (any singleton set is irred)

Def A subset $A \subset X$ is irreducible if $A \neq \emptyset$ & A is irreducible under the induced topology

Def ~~A~~ $A \subset X$ is an irreducible component if A is a maximal irred subset of X

Lemma ① If $Z \subset X$ irred. $\Rightarrow \bar{Z}$ is irred. subset of X

② Any irred. component of X is closed

③ Any irred subset is contained in an irred. component.

④ Any point in X is contained in some irred component.

Proof ① If $\bar{Z} = Z_1 \cup Z_2$, $Z_i \subset \bar{Z}$: closed. (\Rightarrow it is closed in X)

$$Z = (Z \cap Z_1) \cup (Z \cap Z_2)$$

$Z =$ irred subset, so either $\underline{Z \subset Z_1}$ or $Z \subset Z_2$.

$$\bar{Z} \subseteq \bar{Z}_1 = Z_1 \subseteq \bar{Z} \Rightarrow \bar{Z} = Z_1 \therefore \text{it is closed in } X$$

② $Z =$ irred. component of X $Z \subseteq \bar{Z}$. By ① Z is irreducible
 maximality of $Z \Rightarrow Z = \bar{Z} \therefore Z$ is closed in X .

③ $T \subset X$ - irreducible subset.

$$A = \{ T_\alpha \subseteq X : T \subseteq T_\alpha, T_\alpha = \text{irred subset of } X \}$$

$$T \in A \Rightarrow A \neq \emptyset$$

(A, \subseteq) : partially ordered set. $A' \subseteq A$ chain.

$$\text{Let } T' = \bigcup_{\alpha \in A'} T_\alpha$$

Claim T' is an irred subset of X .

Suppose $T' = Z_1 \cup Z_2$: closed in T' . For each $\alpha \in A'$ either

$$T_\alpha \subset Z_1 \text{ or } T_\alpha \subset Z_2 \quad (\because T_\alpha = \text{irred})$$

Suppose $\exists \alpha_0 \in A'$ st. $T_{\alpha_0} \not\subset Z_1$. (if not $T_\alpha \subset Z_1 \forall \alpha \Rightarrow T' \subset Z_1$)

Since A' is a chain

$$T_\alpha \subset Z_2 \forall \alpha \in A' \quad (\text{if } \alpha_1 \leq \alpha_0 \Rightarrow T_{\alpha_1} \subset T_{\alpha_0} \subseteq Z_2 \\ \text{if } \alpha_0 \leq \alpha_2 \Rightarrow T_{\alpha_0} \not\subset Z_1 \text{ otherwise } \\ T_{\alpha_2} \subset Z_1 \Rightarrow T_{\alpha_1} \subset T_{\alpha_2} \subset Z_1 \text{)}$$

$$\Rightarrow T' \subset Z_2 \subset T' \therefore T' = Z_2 \quad \square$$

Any chain of A has an upperbound. By Zorn's lemma, \exists maximal element in A

This is an irred component containing T

④ } x } : \text{irred}

Q2 Y : noetherian (DCC for closed subsets)

$$= Y_1 \cup \dots \cup Y_r \quad \text{closed irreducible components}$$

$$\dim Y = \max \dim Y_i ?$$

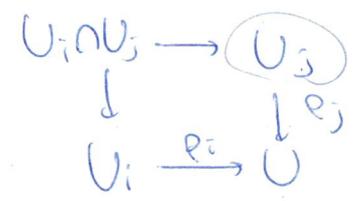
• $\dim Y_i \leq \dim Y$ bc $Y_i = \text{closed}$.

• $\dim Y \leq \max \dim Y_i$ bc any closed $Z \subset Y$ is contained in $\overline{\text{irred}}$ component.

Sheaves Notation $\text{Eq}(X \xrightarrow{f} Y) = \{ x \in X : f(x) = g(x) \}$

X : topological space.

$\text{PreShv}(X)$: category of presheaves (of ~~abelian groups~~ ^{sets})



\uparrow
 $\text{Shv}(X)$: category of sheaves (of ~~abelian groups~~ ^{sets})

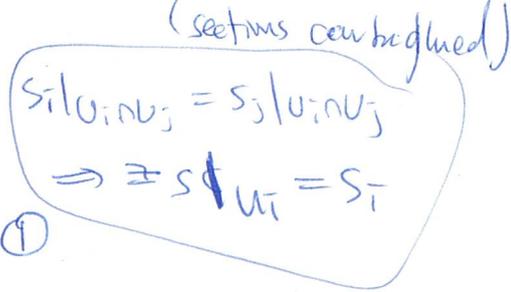
Two additional properties

① $U = \bigcup_{i \in I} U_i, F(U) \rightarrow \prod_{i \in I} F(U_i)$ is injective. (agrees locally \Rightarrow agrees globally)

② $F(U) \rightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j)$ exact i.e. $F(U) \cong \text{Eq}$

Lemma $\text{Shv}(X) \rightarrow \text{PreShv}(X)$ has a left adjoint.

Existence of left adjoints in two steps (Sheafification)



$\text{PreShv}^*(X) =$ category of presheaves which satisfies ①

Step 1 $\text{PreShv}^{(*)}(X) \rightarrow \text{PreShv}(X)$ has a left adjoint.

Want to construct

$$\text{PreShv}(X) \longrightarrow \text{PreShv}^{(*)}(X) \quad F \longmapsto F^s$$

For any $U \subseteq X$ open subset, $F(U)$ equivalence relation on $F(U)$ as follows

$a, b \in F(U)$ is equivalent iff \exists covering $\{U_i \rightarrow U\}_{i \in I}$ st a, b have the same image under

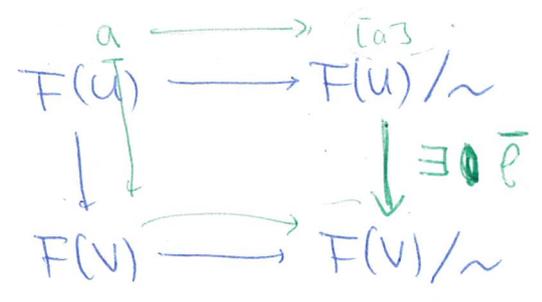
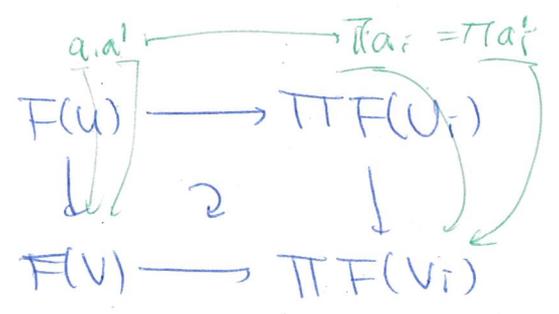
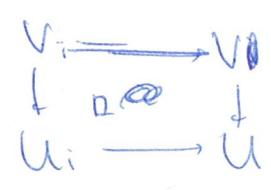
$$F(U) \longrightarrow \prod_{i \in I} F(U_i)$$

This is an equivalence relation bc $a \sim b \wedge b \sim c \Rightarrow a \sim c$
 $a \sim b \Rightarrow b \sim a$

Define $F^s(U) := F(U) / \sim$

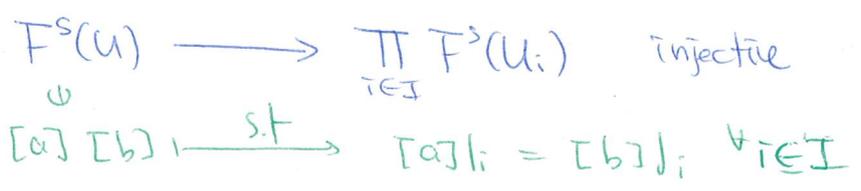
Claim F^s is a presheaf satisfying ①

$$V \hookrightarrow U \text{ open}$$



Well defined by $a, a' \rightarrow [a], \exists U_i \rightarrow U$

Satisfying ①



Choose a lift of $a, b \Rightarrow [a]_i = [a]_i$
 $[b]_i = [b]_i$

$[a]_i = [b]_i$ in $F^s(U_i) \Rightarrow \exists U_{ij} \rightarrow U_i \quad j \in J_i \text{ s.t.}$

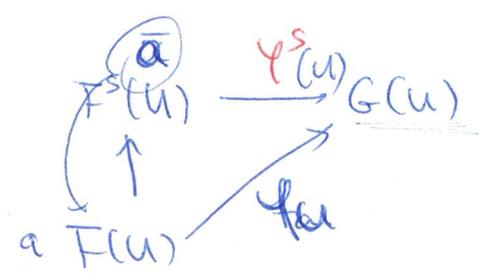
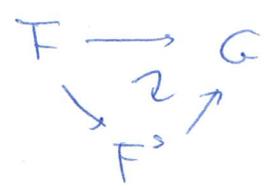


$\Rightarrow F(U) \longrightarrow \prod_{i,j} F(U_{ij}) \quad a, b$ has a same image

$\Rightarrow [a] = [b]$ in $F^s(U)$.

Claim' Suppose G : presheaf satisfying ① $F \xrightarrow{\varphi} G$ morphism in $\text{PreSheaf}(X)$

$\Rightarrow \exists!$



Well defined a, a' : different lifts of $\bar{a} \quad \varphi_U(a) = \varphi_U(a')$

G satisfies ①. so enough to check after refinement. \checkmark

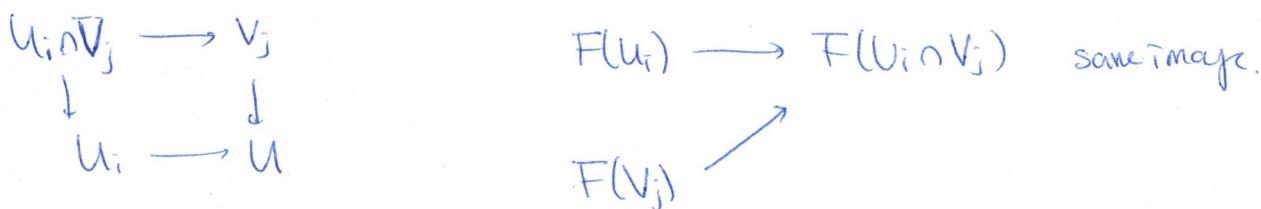
universality (homomorphism) $\Rightarrow \text{Hom}(\dots, (F^s(G))) = \text{Hom}(\dots, (G))$

Step 2 $\text{Shv}(X) \longrightarrow \text{PreShv}^{(*)}(X)$ has a left adjoint.

$U \subset X$ open

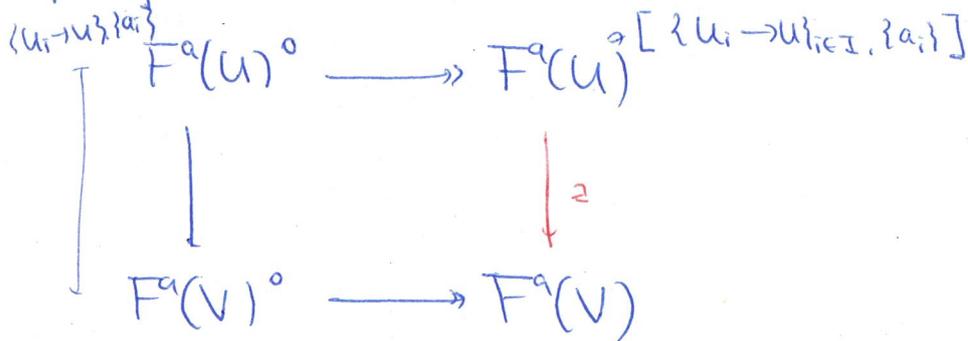
$$F^a(U) = \left\{ \left(\{U_i \rightarrow U\}_{i \in I}, \{a_i\} \right) : \{U_i \rightarrow U\}_{i \in I} \text{ covering} \right. \\ \left. \{a_i\} \in \text{Eq} \left(\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j) \right) \right\} / \sim$$

$$\left(\{U_i \rightarrow U\}_{i \in I}, \{a_i\} \right) \sim \left(\{V_j \rightarrow U\}_{j \in J}, \{b_j\} \right) \text{ iff } \forall a_i, b_j$$



Claim F^a is a sheaf.

• presheaf $V \hookrightarrow U$ open



$\{V_i \rightarrow V\}, \{a_i, b_i\}$