Q1 X : topological space

Def X is reducible if $X = X_1 \cup X_2$, $X_1, X_2 \subset X$ closed and proper ($\neq \phi$ or X).
X is irreducible if it is not reducible.

$\dim X := \sup\{ X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r \}$. $X_i \neq \phi$. closed irred subsets of X.

Claim X : nonempty Hausdorff space. Then $\dim X = 0$

Proof ① Let $X_0 \subset X$ any closed subset with $|X_0| \geq 2$

$x_1 \neq x_2 \in X_0$, nbh $x_1 \in U_1$, $x_2 \in U_2$. $U_1 \cap U_2$.

$\Rightarrow X_0 = U_1^c \cup U_2^c$. $U_1^c, U_2^c \subset X_0$ closed $\therefore$ X_0 is ~~not~~ reducible

~~$\Rightarrow$ ⊗ possible chain $\{x\}$ (any singleton is~~

② Any point $x \in X$ is closed.

$$U := \bigcup_{\substack{y \in X \\ y \neq x}} \underbrace{U_y}_{U_y \cap \{x\} = \phi} = X - \{x\} \text{ is open}$$

$\Rightarrow$ Possible chain $\{x\}$. (any singleton set is irred)

Def A subset $A \subset X$ is irreducible if $A \neq \phi$ & it is irreducible under the induced topology.

Def ~~A~~ ~~An irredu~~ $A \subset X$ is an irreducible component if A is a maximal irred ~~subspace~~ set of X

Lemma ① If $Z \subset X$ irred. $\Rightarrow \bar{Z}$ is irred. subset of X

② Any irred. component of X is closed

③ Any irred subset is contained in an irred. component.

④ Any point in X is contained in some irred component.

<u>Proof</u> ① If $\overline{Z} = Z_1 \cup Z_2$, $Z_i \subset \overline{Z}$: closed. ($\Rightarrow$ it is closed in X)

$Z = (Z \cap Z_1) \cup (Z \cap Z_2)$.

Z = irred subset, so either $\underline{Z \subset Z_1}$ or $Z \subset Z_2$.

$\overline{Z} \subseteq \overline{Z_1} = Z_1 \subseteq \overline{Z} \Rightarrow \overline{Z} = Z_1$ $\therefore$ it is closed in X

② Z = irred. component of X $Z \subseteq \overline{Z}$. By ① $\overline{Z}$ is irreducible

maximality of $Z \Rightarrow Z = \overline{Z}$ $\therefore$ Z is closed in X.

③ $T \subset X$ - irreducible subset.

$A = \{ T_\alpha \subseteq X : T \subseteq T_\alpha,\ T_\alpha = \text{irred subset of } X \}$.

$T \in A \Rightarrow A \neq \phi$.

$(A, \subseteq)$: partially ordered set. $A' \subseteq A$ chain.

Let $T' = \bigcup_{\alpha \in A'} T_\alpha$

<u>Claim</u> T' is an irred subset of X.

Suppose $T' = Z_1 \cup Z_2$: closed in T'. For each $\alpha \in A'$ either

$T_\alpha \subset Z_1$ or $T_\alpha \subset Z_2$ ($\because T_\alpha$: irred)

Suppose $\exists\, \alpha_0 \in A'$ s.t. $T_{\alpha_0} \not\subset Z_1$. (if not $T_\alpha \subset Z_1\ \forall \alpha \Rightarrow T' \subset Z_1$)

Since A' is a chain

$T_\alpha \subset Z_2\ \forall\ \alpha \in A'$ (if $\alpha_1 \leq \alpha_0 \Rightarrow T_{\alpha_1} \subset T_{\alpha_0} \subseteq Z_2$
if $\alpha_0 \leq \alpha_2 \Rightarrow T_{\alpha_2} \not\subset Z_1$ otherwise
$T_{\alpha_2} \subset Z_1 \Rightarrow T_{\alpha_0} \subset T_{\alpha_2} \subset Z_1$ ↯)

$\Rightarrow T' \subset Z_2 \subset T'$ $\therefore T' = Z_2$ ☒

Any chain of A has an upperbound. By Zorn's lemma, $\exists$ maximal element in A

This is an irred component containing T

④ $\{x\}$: irred

<u>Q2</u> Y : noetherian (DCC for closed subsets)

$= Y_1 \cup \cdots \cup Y_r$ ~~closed~~ irred components

$\dim Y = \max \dim Y_i$?

- $\dim Y_i \leq \dim Y$ bc Y_i = closed.
- $\dim Y \leq \max \dim Y_i$ bc any closed $Z \subset Y$ is contained in irred compt.

Sheaves Notation $\mathrm{Eq}(X \underset{g}{\overset{f}{\rightrightarrows}} Y) = \{ x \in X : f(x) = g(x) \}$

X : topological space.

$$
\begin{array}{ll}
\mathrm{PreShv}(X) & : \text{category of presheaves (of } \cancel{\text{abelian groups}} \text{ sets)} \\
\uparrow & \\
\mathrm{Shv}(X) & : \text{category of sheaves (of } \cancel{\text{abelian groups}} \text{ sets)}
\end{array}
$$

$$
\begin{array}{ccc}
U_i \cap U_j & \longrightarrow & U_j \\
\downarrow & & \downarrow \rho_j \\
U_i & \xrightarrow{\rho_i} & U
\end{array}
$$

Two additional properties

① $U = \bigcup_{i \in I} U_i$. $F(U) \longrightarrow \prod_{i \in I} F(U_i)$ is injective. (agrees locally $\Rightarrow$ agrees globally)

② $F(U) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j)$ exact ie $F(U) \xrightarrow{\cong} \mathrm{Eq}$ (sections can be glued)

$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \Rightarrow \exists s \; s|_{U_i} = s_i$

Lemma $\mathrm{Shv}(X) \longrightarrow \mathrm{PreShv}(X)$ has a left adjoint.

Existence of left adjoints in two steps (Sheafification)

$\mathrm{PreShv}^{*}(X)$ = category of presheaves which satisfies ①

Step 1 $\mathrm{PreShv}^{(*)}(X) \longrightarrow \mathrm{PreShv}(X)$ has a left adjoint.

Want to constructed

$$\mathrm{PreShv}(X) \longrightarrow \mathrm{PreShv}^{(*)}(X) \quad F \longmapsto F^s$$

For any $U \subseteq X$ open subset, equivalence relation on $F(U)$ as follows

$a, b \in F(U)$ is equivalent if $\exists$ covering $\{U_i \to U\}_{i \in I}$ st a, b have the same image under

$$F(U) \longrightarrow \prod_{i \in I} F(U_i).$$

This is an equivalence relation bc $a \sim a$ ✓ $a \sim b \Rightarrow b \sim a$, $a \sim b, b \sim c \Rightarrow a \sim c$

Define $F^s(U) := F(U)/\sim$

Claim F^s is a presheaf satisfying ①

$V \hookrightarrow U$ open

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ U_i \arrow[r] & U \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, mapsto] & {[a]} \\ F(U) \arrow[r] \arrow[d] & F(U)/\sim \arrow[d, "\exists\, \bar{\rho}"] \\ F(V) \arrow[r] & F(V)/\sim \end{tikzcd}$$

$$\begin{tikzcd} a, a' \arrow[r, mapsto] & \prod a_i = \prod a_i' \\ F(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \prod F(U_i) \arrow[d] \\ F(V) \arrow[r] & \prod F(V_i) \end{tikzcd}$$

Well-defined b/c

$a, a' \longrightarrow [a]$, $\exists U_i \to U$

Satisfying ①

$$F^s(U) \longrightarrow \prod_{i \in I} F^s(U_i) \quad \text{injective}$$

$[a]\ [b] \longmapsto$ s.t. $[a]|_i = [b]|_i \ \forall i \in I$

Choose a lift of $a, b \Rightarrow [a]|_i = [a|_i]$

$[b]|_i = [b|_i]$

$[a_i] = [b_i]$ in $F^s(U_i) \Rightarrow \exists\, U_{ij} \to U_i \quad j \in J_i$ s.t.

$a|_{U_{ij}} = b|_{U_{ij}}$.

$\Rightarrow \quad F(U) \longrightarrow \prod_{i,j} F(U_{ij}) \quad a, b$ has a same image

$\Rightarrow [a] = [b]$ in $F^s(U)$.

Claim' Suppose G: presheaf satisfying ① $F \xrightarrow{\varphi} G$ morphism in $PreSh(X)$

$\Rightarrow \exists!$

$$\begin{tikzcd} F \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & G \\ & F^s \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} \bar{a} \in F^s(U) \arrow[r, "\varphi^s(U)"] & G(U) \\ a \in F(U) \arrow[u] \arrow[ur, "\varphi_U"'] & \end{tikzcd}$$

Well defined a, a': different lifts of $\bar{a}$ $\varphi_U(a) \overset{?}{=} \varphi_U(a')$

G satisfies ①. so enough to check after refinement. ✓

uniqueness Obvious $\Rightarrow Hom_{PreSh(X)}(F^s, G) = Hom_{PreSh}(F, G)$

Step 2 $Shv(X) \longrightarrow PreShv^{(*)}(X)$ has a left adjoint.

$U \subseteq X$ open

$$F^a(U) = \left\{ \left(\{U_i \to U\}_{i\in I}, \{a_i\}\right) : \begin{array}{l} \{U_i \to U\}_{i\in I} \text{ covering} \\ \{a_i\} \in Eq\left(\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)\right) \end{array} \right\} \Big/ \sim$$

$$\left(\{U_i \to U\}_{i\in I}, \{a_i\}\right) \sim \left(\{V_j \to U\}_{j\in J}, \{b_j\}\right) \text{ iff } \forall\, a_i, b_j$$

$$\begin{tikzcd} U_i \cap V_j \arrow[r] \arrow[d] & V_j \arrow[d] \\ U_i \arrow[r] & U \end{tikzcd} \qquad \begin{tikzcd} F(U_i) \arrow[r] & F(U_i \cap V_j) \\ F(V_j) \arrow[ur] & \end{tikzcd} \quad \text{same image.}$$

Claim F^a is a sheaf.

- presheaf $V \hookrightarrow U$ open

$$\begin{tikzcd} \{U_i \to U\}\{a_i\} \arrow[ddd, maps to] & F^a(U)^\circ \arrow[r, two heads] \arrow[d] & F^a(U) \ni [\{U_i \to U\}_{i\in I}, \{a_i\}] \arrow[d, "\exists"] \\ & F^a(V)^\circ \arrow[r] & F^a(V) \\ \\ \{V_i \to V\}, \{a_i|_{V_i}\} & & \end{tikzcd}$$