Notion of colimit (direct limit)

$(I, \leq)$: directed set

- $a \leq a$
- $a \leq b,\ b \leq c \Rightarrow a \leq c$
- $a, b \in I,\ \exists c$ s.t $a \leq c$, and $b \leq c$

$\rightsquigarrow$ Think of it as a category.

$\mathrm{Hom}(a, a) = \{id_a\}$

$\mathrm{Hom}(a, b) = \{*\}$ if $a \leq b$, ϕ o.w

$\mathcal{C}$: category.

Def A direct system over I is a functor

$$A : I \longrightarrow \mathcal{C}.$$

Def (colimit) Let $A : I \to \mathcal{C}$ a directed system

Its colimit

$$\operatorname*{colim}_I A_i = \coprod_{i \in I} A_i / \sim \qquad \text{where}$$

$x_i \sim x_j$ iff $\exists k$. $i \leq k,\ j \leq k$ s.t

$$\begin{tikzcd} A_i \arrow[r, "f_{ik}"] & A_k \\ A_j \arrow[ur, "f_{jk}"'] & \end{tikzcd} \qquad f_{ik}(x_i) = f_{jk}(x_j).$$

Universal property $\exists\ \phi_i : A_i \longrightarrow \operatorname{colim} A_i$

If $\{\psi_i : A_i \longrightarrow B\}$ s.t $i \leq j$

$$\begin{tikzcd} A_i \arrow[rr, "f_{ij}"] \arrow[dr] \arrow[ddr, bend right, "\psi_i"'] & & A_j \arrow[ddl, bend left, "\psi_j"] \\ & \operatorname{colim} A_i \arrow[d, "\Psi\ \exists !"] & \\ & B & \end{tikzcd}$$

$[x, i] = [y, j]$ in $\operatorname*{colim}_I A_i$

$\Rightarrow \exists\ z \in A_k$ s.t

$[x, i] = [y, j] = [z, k]$

$$\mathrm{Hom}(\operatorname*{colim}_I A_i, B) = \lim_I \mathrm{Hom}(A_i, B).$$

Direct / Inverse Image

$f : X \longrightarrow Y$: continuous map between top spaces

$f_* : Shv(X) \longrightarrow Shv(Y)$ direct image functor

$f^{-1} : Shv(Y) \longrightarrow Shv(X)$ inverse image functor

Def $\mathcal{F} \in Shv(X)$. $f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}V)$, $V \subseteq Y$ open.

Prop $f_*\mathcal{F}$ is a sheaf. & f_* a functor.

Ex $f : X \longrightarrow pt$ $\quad f_*\mathcal{F} = \mathcal{F}(X)$

Def. $\mathcal{G} \in Shv(Y)$, $f^{\#}\mathcal{G}(U) := \operatorname{colim}_{f(U) \subseteq V} \mathcal{G}(V)$ ← presheaf

- $f^{-1}\mathcal{G} := (f^{\#}\mathcal{G})^a$.

Ex $f : x \longrightarrow X$ point. $\quad f^{-1}\mathcal{F} =: \mathcal{F}_x$ (stalk at x)

Ex 2 $f : pt \sqcup pt \longrightarrow pt$. Compare $f^{\#}\underline{\mathbb{C}}_{pt}$ vs $f^{-1}\underline{\mathbb{C}}_{pt}$

Prop $f^{-1} \dashv f_*$

$$Hom_{Shv(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = Hom_{Shv(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

Prop f^{-1} is exact.

<u>Lemma</u> X : top sp. $f : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Shv}(X)$.

f is an isomorphism $\iff$ $\forall x \in X$. $f_x : \mathcal{F}_x \to \mathcal{G}_x$ isom

Proof] ($\Rightarrow$) Clear

($\Leftarrow$) $U \subset X$ open. enough to show $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ isom.

<u>Claim 1</u> $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective

pf) $s, t \in \mathcal{F}(U)$ s.t $s_x = t_x$ $\forall x \in U$ $\exists$ $V_x \subset U$ nbh of x s.t

$s|_{V_x} = t|_{V_x}$

Consider $\mathcal{F}(U) \hookrightarrow \prod_{x \in U} \mathcal{F}(V_x)$ $\Rightarrow s = t$ ☒

$\Rightarrow$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f(U)"] \arrow[d, hook] & \mathcal{G}(U) \arrow[d, hook] \\ \prod_{x \in U} \mathcal{F}_x \arrow[r, "\cong"] & \prod_{x \in U} \mathcal{G}_x \end{tikzcd}$$

$\Rightarrow f(U)$ injective #

<u>Claim 2</u> $\{s_x\} \in \prod_{x \in U} \mathcal{F}_x$. Suppose $\exists \{U_i \to U\}_{i \in I}$

and $\mathcal{F}(U_i) \ni f_i$ s.t $(f_i)_y = s_y$ $\forall y \in U_i$

$\Longrightarrow \exists f \in \mathcal{F}(U)$ s.t $f_i = f$.

<u>Proof</u>

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ (bc <u>claim 2</u>) ☒

Proof of surjectivity

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f(U)"] \arrow[d, hook] & \mathcal{G}(U) \arrow[d, hook] \ni s \\ \prod \mathcal{F}_x \arrow[r, "\cong"] & \prod_{x \in U} \mathcal{G}_x \end{tikzcd} \qquad s \mapsto (s_x)$$

$\exists$ nbh U_x of x, $T(x) \in \mathcal{F}(U_x)$ s.t.

$$f_x(T(x)_x) = s_x. \qquad (\text{bc } f_x \text{ surj})$$

$\Rightarrow$ Further shrink U_x, may assume

$$f(U_x)(T(x)) = s|_{U_x}$$

Consider $\{U_x \to U\}_{x \in U}$ and $T(x) \in \mathcal{F}(U_x)$

$$(T(x))_y = T(x)_x$$

$\Rightarrow$ By Claim 2 ☑

$\boxed{\text{Back to Q5}}$

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "f"] \arrow[dr] & & \mathcal{C} \\ & \mathcal{B}^a \arrow[ur, "f^a"'] & \end{tikzcd}$$

Check fiberwise isom for f^a_x, $x \in X$. ☑

We missed the following def. $(X, \mathcal{O}_X)$: ringed space

<u>Def</u> $(f, f^\#) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ morphism consists of

- $f : X \to Y$ a continuous map
- $f^\# : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ in $\mathrm{Shv}(Y)$

Exercise 3

R_X : sheaf of continuous maps on X.

Recall $X \overset{f}{\hookrightarrow} \mathbb{A}^n_k$: closed algebraic subset.

$k[X] := k[x_1 \cdots x_n] / I(X)$.

$U \subseteq X$ open. $f : U \to k$. Then f is regular at p if $\exists$ nbh V of p in U s.t

$$f|_V = \frac{g}{h} \qquad g.h \in k[X].$$

Define the structure sheaf

$$\mathcal{O}_X(U) = \{ f : U \to k, \ f \text{ is regular at } p \in U \}.$$

Def A k-ringed space $(X. \mathcal{O}_X)$ is an affine variety if

$$(X. \mathcal{O}_X) \cong (W, \mathcal{O}_W), \qquad W \subseteq \mathbb{A}^n$$

Lemma Let $(X. \mathcal{O}_X)$: affine variety. $Z \subseteq X$ a closed subset, $p \in Z$ and $f : U \to k$ a function defined on a neighborhood U in Z of p.
Then f is regular at p iff $\exists$ nbh V of p in X and $g \in \mathcal{O}_X(V)$ s.t

$$f|_{V \cap U} = g|_{V \cap U}.$$

Proof ($\Leftarrow$) : Clear (restriction of regular is regular)

($\Rightarrow$) Further shrink U, $f = \frac{r}{s}$, $r.s \in k[Z]$.

Since $k[X] \twoheadrightarrow k[Z]$, $\exists \tilde{r}. \tilde{s} \in k[X]$ which map to $r.s$.

Let $V = D(\tilde{s}) \subset X$ open. $g = \frac{\tilde{r}}{\tilde{s}} \in \mathcal{O}_X(V)$. $\boxtimes$

Let $f : X \hookrightarrow \mathbb{A}^n$ a closed subset. Then

$$\begin{tikzcd} \varphi_a : f^{\#}\mathcal{O}_{\mathbb{A}^n} \arrow[r] & R_X \\ f^{-1}\mathcal{O}_{\mathbb{A}^n} \arrow[u, hook] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Note $\varphi : \mathcal{F} \to \mathcal{G}$

$\operatorname{Im}\varphi = (\operatorname{Im}(\varphi(U)))^a$.

<u>Claim</u> $\operatorname{Im}\varphi \cong \mathcal{O}_X$.

Pf) Enough to check it is an isom on each stalk. $x \in X$

$$\varphi_x : (f^{-1}\mathcal{O}_{\mathbb{A}^n})_x \longrightarrow \mathcal{O}_{X,x}$$

$$(f^{-1}\mathcal{O}_{\mathbb{A}^n})_x = (f^{\#}\mathcal{O}_{\mathbb{A}^n})_x \qquad \mathcal{O}_{X,x} \ni$$

$$[(V,s)] \qquad\qquad [(U,t)]$$

This follows from the above lemma. ☐

<u>Cor</u> $(X,\mathcal{O}_X)$: algebraic variety $Z \subseteq X$ closed subset.

Let $\mathcal{O}_Z = \underline{\operatorname{Im}(f^{-1}\mathcal{O}_X \longrightarrow R_Z)}$. Then $(Z, \mathcal{O}_Z)$ is an algebraic variety (this is precisely the sheaf used in Q7).

<u>Proof</u> $X = \bigcup_{i=1}^{n} X_i$: affine chart.

$$\begin{tikzcd} \underline{Z \cap X_i} \arrow[r, hook, "f", "\text{closed}"'] \arrow[d, "q"'] & X_i \arrow[r, hook, "j"] \arrow[d, hook, "q_i"] & \mathbb{A}^n \\ Z \arrow[r, "f"] & X & \end{tikzcd}$$

$$
\begin{aligned}
q^{-1}\mathcal{O}_Z &= \mathrm{Im}\left(q^{-1}f^{-1}\mathcal{O}_X \longrightarrow q^{-1}R_Z \right) \quad (\text{open}) \\
&= \mathrm{Im}\left(f^{-1}q_i^{-1}\mathcal{O}_X \longrightarrow R_{Z\cap X_i} \right) \\
&= \mathrm{Im}\left(f^{-1}\mathcal{O}_{X_i} \longrightarrow R_{Z\cap X_i} \right) \\
&= \mathrm{Im}\left(f_i^{-1}j^{-1}\mathcal{O}_{\mathbb{A}^n} \longrightarrow R_{Z\cap X_i} \right) \quad (\text{cor}) \\
&= \mathcal{O}_{Z\cap X_i}
\end{aligned}
$$

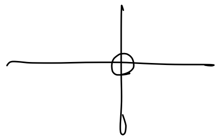
$\square$

Ex2. $X = \mathbb{A}^2 \quad U = X - \{(0,0)\} \quad j : U \hookrightarrow X$

$$j^* : \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U)$$

<u>Claim</u> j^* is an isom.

<u>Surjective</u>: $U = D(x) \cup D(y)$.



Let $\varphi \in \mathcal{O}_X(U)$.

$$\varphi|_{D(x)} \in k[x, x^{-1}, y] \Rightarrow \varphi|_{D(x)} = \frac{f}{x^m}, \ f \in k[x,y]$$

$$\varphi|_{D(y)} \in k[y, y^{-1}, x] \Rightarrow \varphi|_{D(y)} = \frac{g}{y^n} \quad g \in k[x,y]$$

$$\frac{f}{x^m} = \frac{g}{y^n} \quad \text{on } D(x) \cap D(y) = D(x,y)$$

$$\Rightarrow \frac{f}{x^m} = \frac{g}{y^n} \quad \text{in } k[x, x^{-1}, y, y^{-1}]$$

$$\Rightarrow y^n f = x^m g \quad \text{in} \quad k[x,y]$$ $(x,y) = 1$

$k[x,y]$: UFD, $\quad x^m | f, \ y^n | g$

$f = x^m f_0, \ g = y^n g_0, \ f_0, g_0 \in k[x,y]$

$\Rightarrow f_0 \longmapsto \varphi$ under j^*

<u>Injective</u>:

$$\begin{tikzcd} \mathcal{O}_X(U) \arrow[r, hook] & \mathcal{O}_X(D(x)) \times \mathcal{O}_X(D(y)) \\ \mathcal{O}_X(X) \ni f \arrow[u, "j^*"] \arrow[ur] & \end{tikzcd}$$

Injective bc $k[x,y]$ domain.

$\blacksquare$