Ex 4 $\quad k = \overline{k}$

Q1 X = irreducible $\subseteq \mathbb{A}^n_k$.

$k[X]$ = domain

$k(X) := \mathrm{Frac}(k[X])$

Claim

$$k(X) \simeq \left\{ (U, \varphi) : \begin{array}{l} U \subseteq X \text{ nonempty open} \\ \varphi : U \longrightarrow \mathbb{A}^m_k \text{ regular } ~~fcn~~ \end{array} \right\} \Big/ \sim$$

$(U_1, \varphi_1) \sim (U_2, \varphi_2)$ iff $\varphi_1 = \varphi_2$ on $U_1 \cap U_2$

For simplicity A = set on the RHS.

Normal form : $x \in U$, φ : regular at $x \Rightarrow \varphi = \frac{g}{h}$, $g, h \in k[X]$.

$$(U, \varphi) \sim (D(h), \varphi|_{D(h)}) \sim (D(h), \tfrac{g}{h})$$

A : field bc $(D(h), \frac{g}{h}) \cdot (D(g), \frac{h}{g}) = (D(h) \cap D(g), 1)$

Define a ring map

$$\alpha : k(X) \longrightarrow A$$
$$\frac{g}{h} \longmapsto (D(h), \tfrac{g}{h})$$

Injective bc $k(X)$, A field.

Surjective : by normal form.

Q2 $L_1, L_2 \subset \mathbb{P}^2_k$ lines $[x_0 : x_1 : x_2]$

$L_1 \neq L_2$

3

Step 1 $L_1 \cap L_2 \neq \emptyset$

Suppose $L_1 \cap L_2 = \emptyset$

$\mathrm{Aut}(\mathbb{P}^2_k) \supset PGL_2(k)$

Move $L_1 = V(x_0)$

$\mathbb{P}^2_k = \mathbb{P}^1_k \sqcup \mathbb{A}^2_k$

$\Rightarrow L_2 \subseteq \mathbb{A}^2_k \iff k[x_1, x_2] \longrightarrow \mathcal{O}_{L_2}(L_2) \cong k$

$\therefore L_2 \longrightarrow \mathbb{A}^2_k$ constant $\lightning$

$\therefore L_1 \cap L_2 \neq \emptyset$

Step 2 $|L_1 \cap L_2| = 1$

O.w $P_1 \neq P_2 \in L_1 \cap L_2$

$\Rightarrow \overline{P_1 P_2} \subseteq L_1 \cap L_2$ $\lightning$

Q3 $k = \overline{\mathbb{F}_p}$. Frob : $\mathbb{A}^n_k \longrightarrow \mathbb{A}^n_k$

$(a_0 \cdots a_n) \longmapsto (a_0^p, \cdots a_n^p)$

Def k : perfect if every irred poly over k has distinct roots.

Equivalenty : char$(k)=0$ or char$(k)=p$ and $x \mapsto x^p$ is an aut. of k.

Any algebraically closed field is perfect.

So Frob induces bijection on k - points.

Why not isom ?

Frob : isom $\Longleftrightarrow$ $k[x_1, \cdots x_n] \longrightarrow k[x_1, \cdots x_n]$

$x_i \longmapsto x_i^p$

isom

Clearly not isom by degree of variables.

- Fixed point

$\mathbb{F}_{p^m}$ = Splitting field of $X^{p^m} = X$ over $\mathbb{F}_p$

$\subseteq \overline{\mathbb{F}_p}$

$\Rightarrow$ fixed points defined over $\mathbb{F}_{p^m}$.