Q1 Def of sheaf X : top sp. $\mathcal{F} \in Shv(X)$.

a/ $U \subseteq X$ open $\quad U_\circ = \bigcup_{i \in I} U_i$ open

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \qquad \text{inject}$$

b/ $$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \quad \text{exact.}$$

Let's compare defn in the class. $s, t \in \mathcal{F}(U)$ st

- $s|_{U_i} = t|_{U_i} \ \forall i \overset{a/}{\Longrightarrow} \text{as } s = t \in \mathcal{F}(U)$
- $s_i \in \mathcal{F}(U_i)$ st $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \overset{b/}{\Longrightarrow} \exists\, s \in \mathcal{F}(U)$ st $s|_{U_i} = s_i$ $\quad \boxed{/\!/}$

Q2 Sheafification : $i : Shv(X) \longrightarrow PreShv(X)$ inclusion.

$(\ )^a : PreShv(X) \longrightarrow Shv(X)$. left adjoint to i. so

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[d, "\alpha(*)"'] & G \\ \mathcal{F}^a \arrow[ur, "\exists!\, f^a"'] & \end{tikzcd}$$

$(*)$ is from

$Id \in Hom_{Shv(X)}(\mathcal{F}^a, \mathcal{F}^a) = Hom_{PreShv(X)}(\mathcal{F}, i\mathcal{F}^a)$

Eg $$\mathcal{F}(U) = \begin{cases} \mathbb{Z} & U \subseteq X \text{ nonempty} \\ 0 & U = \phi \ (*) \end{cases}$$

$U \hookrightarrow V \Rightarrow \mathcal{F}(V) \xrightarrow{=} \mathcal{F}(U)$ identity map if $\cdot\, U \neq \phi,\ V \neq \phi$

$\cdot\, U = V = \phi$

This is a presheaf of abelian groups on X.

It is not a sheaf bc it does not satisfy b/. In general

If X : Hausdorff. $|X| \geq 2$. Choose $U = U_1 \sqcup U_2$, $U_i \neq \emptyset$.

$$F(U) \longrightarrow F(U_1) \times F(U_2) \rightrightarrows F(U_1 \cap U_2) \quad \text{not exact}$$

$$\mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z} \rightrightarrows 0$$

$$x \longmapsto (x, x)$$

Why (*)? $\mathcal{F}(\emptyset)$ of a sheaf $\mathcal{F}$ should be always the terminal object of your category.

Taking sheafification,

$$F^a(U) = \{ U \longrightarrow \mathbb{Z} \text{ locally const} \}. \quad (\text{or } \underline{\mathbb{Z}}_X)$$

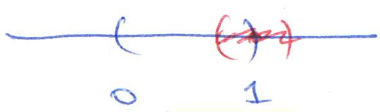
Check Explicity compute F^a using the definition.

Q3 Direct / Inverse image functor

Example 1 $f : X \hookrightarrow Y$ subset w/ induced topology Then

$$f_* \underline{\mathbb{Z}}_X = \underline{\mathbb{Z}}_{\bar{X}} \quad \text{closure inside } Y$$

Why $\bar{X}$?



$Y = \mathbb{R}$

Example 2 • $Y = S^1$. $X_1 = S^1 \sqcup S^1$.

$$f_* \underline{\mathbb{Z}}_{X_1} \cong \underline{\mathbb{Z}}_Y \oplus \underline{\mathbb{Z}}_Y$$

• $Y = S^1$. $X_2 =$ the Möbius band

$$f_* \underline{\mathbb{Z}}_{X_2} \neq f_* \underline{\mathbb{Z}}_{X_1} \quad \text{but fiberwise rank} = 2$$

$f_* \underline{\mathbb{Z}}_{X_2}(Y) = \mathbb{Z}$ bc X_2 connected

Inverse image $f : X \longrightarrow Y$

Example 1 $f : * \longrightarrow Y$. $f^{-1}\mathcal{F} = \mathcal{F}_x$

Example 2 $f^{-1}\underline{\mathbb{Z}}_Y \simeq \underline{\mathbb{Z}}_X$

Q4 Continuous fcn btw affine variety which is not polynomial

A $\mathbb{A}^1_k \setminus 0 \longrightarrow \mathbb{A}^1_k \qquad x \longmapsto \frac{1}{x}$

Q5 Product of alg varieties : enough to know affine case + gluing

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"] & Y \\
& X &
\end{tikzcd}$$

Construction $X = \bigcup U_i \quad Y = \bigcup V_j$ affine charts $\rightsquigarrow$ Glue $U_i \times V_j$

Locally what happens? $U_i \subset \mathbb{A}^{n_i} \quad V_j \subset \mathbb{A}^{n_j} \Rightarrow U_i \times V_j \subset \mathbb{A}^{n_i + n_j}$

How we glue $\{U_i \times V_j\}_{\substack{i \in I \\ j \in J}}$?

Gluing data

- $\{X_i\}_{i\in I}$: finite collection of alg variety
- $X_{ij} \subset X_i$: collection of open subsets $j \in I$
- $\varphi_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$: isom

] st

a/ $X_{ii} = X_i \qquad \varphi_{ii} = id_{X_i}$

b/ $\varphi_{ij}(X_{ij} \cap X_{ik}) = X_{jk} \cap X_{ji}$ in X_{ji}

c/ $\varphi_{jk}|_{X_{jk}\cap X_{ji}} \circ \varphi_{ij}|_{X_{ij}\cap X_{ik}} = \varphi_{ik}|_{X_{ik}\cap X_{ij}}$

<u>Prop</u> $\exists$ alg variety X equipped with open covering $U_i \subset X$ and isom $\varphi_i : X_i \xrightarrow{\simeq} U_i$ st $\forall$ alg variety Y

$$Hom(X, Y) = \{ (f_i) \in \prod Hom(X_i, Y) : f_j|_{X_{ji}} \circ \varphi_{ij} = f_i|_{X_{ij}} \ \forall i,j \in I \}$$

$$f \longmapsto f_i := f|_{U_i} \circ \varphi_i$$