Q1 $k = \mathbb{C}$. Continuous but not regular map btw $\mathbb{A}^1_{\mathbb{C}}$.

Pf. Consider $f : \mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{C}}$, $z \longmapsto \bar{z}$

f is continuous bc $f^{-1}(\text{closed}) = \text{closed}$.

We show f is not regular by contradiction. Suppose $\exists\, \phi \neq U \subset \mathbb{A}^1_{\mathbb{C}}$ open s.t $f|_U$: rational. ie

$$\bar{z} = \frac{c \prod_{l=1}^{n} (z - \alpha_l)}{\prod_{j=1}^{m} (z - \beta_j)} \quad \text{for some } \alpha_l, \beta_j \in \mathbb{C} \text{ on } U = \mathbb{C} \setminus \{\beta_1 \cdots \beta_m\}$$

$$\bar{z} \prod_{j=1}^{m} (z - \beta_j) = c \prod_{l=1}^{n} (z - \alpha_l)$$

- Take $z \in \mathbb{R}$, $z \to +\infty$; $\bar{z} z^m + \ = c \cdot z^n + \ \Rightarrow \underline{c = 1,\ m = n-1}$
- Take $z \in i\mathbb{R}$, $z \to +i\infty$; $z = i\theta$, $\theta \to +\infty$

$$-i\theta \cdot (i\theta)^m + \ = (i)^n \theta^n + \ \quad \lightning$$

This cannot happen

Q2 X, Y : alg variety. $\dim(X\times Y) = \dim X + \dim Y$

We know : $\cdot$ $X = \bigcup X_\alpha \Rightarrow \dim X = \sup_\alpha \dim X_\alpha$

$\cdot$ equality when X, Y are affine

Take aff covers $X = \bigcup_\alpha X_\alpha$, $Y = \bigcup_\beta Y_\beta$, finite index. (qc)

$$\Rightarrow X\times Y = \bigcup_{\alpha,\beta} X_\alpha \times Y_\beta$$

$$\begin{aligned}\dim X\times Y &= \sup_{\alpha,\beta} \dim(X_\alpha \times Y_\beta)\\ &= \sup_{\alpha,\beta} (\dim X_\alpha + \dim Y_\beta)\\ &= \sup_\alpha \dim X_\alpha + \sup_\beta \dim Y_\beta = \dim X + \dim Y \qquad \square\end{aligned}$$

Q3 $f : X \to Y$ finite $\Rightarrow$ $|f^{-1}(y)| < \infty$

a/ f: affine

b/ $\forall V \subseteq Y$ affine open. $f^* : A(V) \to A(f^{-1}V)$ f.g. $A(V)$-module

May assume X, Y: affine $f^* : A(Y) \to A(X)$.

$y \in Y \Leftrightarrow$ maximal ideal $m_y \subset A(Y)$

$$f^{-1}(m_y) = \{ m_x \subset A(X) : \text{max. ideal} : (f^*)^{-1} m_x = m_y \}$$
$$\overset{(*)}{=} \{ m_x \subset A(X) \quad '' \quad : A(X) \cdot f^*(m_y) \subseteq m_x \}$$

$(*)$ if $(f^*)^{-1}(m_x) = m_y$ then

$$A(X) f^*(m_y) = A(X) \cdot f^*(f^*)^{-1}(m_x) \subseteq m_x$$

if $A(X) \cdot f^*(m_y) \subseteq m_x$ then

$(f^*)^{-1}(m_x) \supseteq (f^*)^{-1}(A(X) f^*(m_y)) \supseteq m_y$. $(f^*)^{-1}(m_x)$ not $A(Y)$ $\Rightarrow$ equal to m_y.

So, $k \cong A(Y)/m_y \longrightarrow \underbrace{A(X)/A(X) \cdot f^*(m_y)}_{\text{f.dim } k\text{-v.sp}}$

Claim If a k-alg R is f.d k-v.sp then $\exists$ finitely many max ideal.

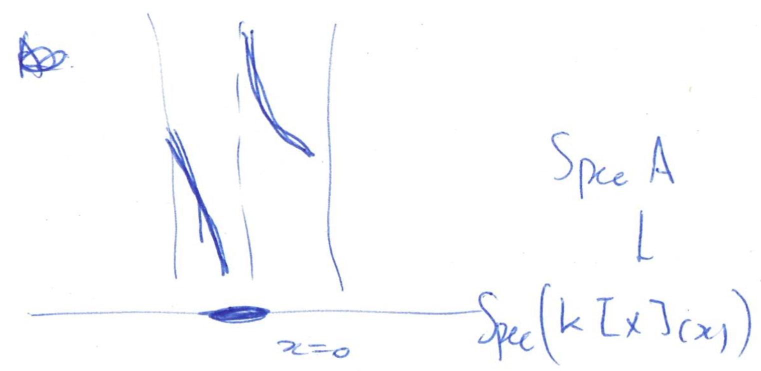
Pf) $m_1, \cdots m_\ell \subset R$ max ideal distinct.

$$R \longrightarrow \prod_{i=1}^{\ell} R/m_i \qquad \text{surjective (CRT)}$$

$$\Rightarrow \infty > \dim_k(R) \geq \sum_{i=1}^{\ell} \dim_k(R/m_i) \geq \ell \qquad \boxtimes$$

$\underline{Q4}$ $A = k[X]_{(x)}[t] \Rightarrow \exists$ codim $=1$ prime ideal $\mathfrak{p} \subseteq A$ s.t. $\dim \mathfrak{p} = 0$

$\dim A = \dim k[X]_{(x)} + 1 = 2$



Take $\mathfrak{p} = (xt-1) \subset A$

A : domain, $\Rightarrow$ $xt-1$ not zero divisor nor unit

Krull's Hauptidealsatz $\Rightarrow$ $\mathrm{ht}(\mathfrak{p}) = 1$ $\therefore$ $\mathrm{codim}(\mathfrak{p}, A) = 1$

<u>Claim</u> $\mathfrak{p} \subset A$ maximal ideal hence $A/\mathfrak{p}$ is a field

$$\begin{array}{ccc} \varphi : k[X]_{(x)}[t] & \longrightarrow & k(x) \\ t & \longmapsto & \frac{1}{x} \\ \downarrow & \nearrow_{\bar{\varphi}} & \\ A/\mathfrak{p} & & \end{array}$$

- $\bar{\varphi}$: surjective
- $\bar{\varphi}$ injective : $\bar{a} \in A/\mathfrak{p}$ s.t. $\bar{\varphi}(\bar{a}) = 0$

$\bar{a} = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n$, $\alpha_i \in k[x]_{(x)}$

$Ma = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n$ $\quad \alpha_i \in \underline{\underline{k[x]}}$

PID $\Rightarrow$ UFD

$\bar{\varphi}(\bar{a}) = \alpha_0 + \alpha_1 \frac{1}{x} + \cdots + \alpha_n \frac{1}{x^n}$ $\quad$ $xt-1$ is irred in $k[x][t]$ (primitive, irred in $k(x)[t]$)

$\Rightarrow x^n \alpha_0 + x^{n-1}\alpha_1 + \cdots + \alpha_n = 0$

$xt-1 \mid a$ in $\mathrm{Frac}(k[x])[t] \Rightarrow xt-1 \mid a$ in $(k[x])[t]$

$\blacksquare$