Q1 X : alg variety / k $\quad p \in X$

$$d_p : \mathcal{O}_{X,p} \longrightarrow m_p / m_p^2 \qquad f \longmapsto f - f(p)$$

Then d_p satisfies the Leibniz rule.

Proof) $d_p(f_1 f_2) = f_1 f_2 - (f_1(p) f_2(p))$

$$= (f_1 - f_1(p)) f_2 + f_2 \cdot f_1(p) - f_1(p) \cdot f_2(p)$$

$$= (f_1 - f_1(p)) f_2 + f_1 (f_2 - f_2(p)) - \underbrace{(f_1 - f_1(p))(f_2 - f_2(p))}_{m_p^2}$$

$$= d_p(f_1) \cdot f_2 + f_1 \, d_p(f_2) \quad (\mathrm{mod}\ m_p^2).$$

$\boxtimes$

$\mathrm{Char}(k)=p\geq 0$

Q2 $C=V(\underbrace{X^m-Y^n}_{=F})\subset \mathbb{A}^2_k$. Find $\mathrm{Sing}(C)$

Krull's Hauptidealsatz any irred component has $\dim = 1$

Case 1) $n=1$ or $m=1$ $\quad k[X,Y]/(X^m-Y)\cong k[X]$: nonsingular

$p\in S$: singular iff $\dim_k \mathfrak{m}_p/\mathfrak{m}_p^2>1$.

Case 2) $n, m\geq 2$ $\quad p=(a,b)\in C$ ie $a^m-b^n=0$

$$\mathfrak{m}_p/\mathfrak{m}_p^2=\frac{(X-a,Y-b)+F}{((X-a),(Y-b))^2+F}$$

$X-a=:u$

$Y-b=:v$

$$=\frac{(u,v)}{(u^2,uv,v^2+(u+a)^m-(v+b)^n)}$$

$$=\frac{(u,v)}{(u^2,uv,v^2,a^m+mua^{m-1}-b^n-nvb^{n-1})}$$

$a^m-b^n=0$

$$=\frac{(u,v)}{(u^2,uv,v^2,mua^{m-1}-nvb^{n-1})}$$

If $m\cdot a^{m-1}\neq 0$, $\mathfrak{m}_p/\mathfrak{m}_p^2=(u,v)/(u^2,uv,v^2,u-\varepsilon v)$ $\quad \varepsilon=(ma^{m-1})^{-1}nb^{n-1}$

$=(u,v)/(\varepsilon^2v^2,\varepsilon v,v^2)$

$=(v)/(v^2)$: 1-dim $\therefore$ regular

$(ma,nb)\neq(0,0)$ $\quad$ $\boxed{\text{If } (a,b)\neq(0,0) \text{ and } p\nmid m \text{ and } n \Rightarrow \text{regular}}$

If $m\cdot a^{m-1}=n\cdot b^{n-1}=0\Rightarrow \dim_k \mathfrak{m}_p/\mathfrak{m}_p^2=2$ $\quad X^p=0$

Goal When $a^m-b^n=ma^{m-1}=nb^{n-1}=0$? $\quad p\mid m,n$

$m=p^s m_0$

$n=p^t n_0$ $\quad s,t\geq 1$

wlog $s\geq t$

$$a^m-b^n=(a^{p^{s-t}m_0}-b^{n_0})^{p^t}$$

$$a^m-b^n=0\iff a^{p^{s-t}m_0}-b^{n_0}=0$$

Replace (m,n) by $(p^{\circ}m_0,n_0)$ $\quad (n_0,p)=1\Rightarrow$ only when $(a,b)=(0,0)$

If $n_0=1$: regular

or $p^{\circ}m_0=1$

$p \nmid m_0, n_0 \geq 2$, $\underline{(p, n_0) = 1}$

If $\gcd(m_0, n_0) = 1 \Rightarrow X^{p^0 m_0} - Y^{n_0}$ is Irred in $k[X][Y]$

(by Gauss)

$\therefore (0,0)$ singular point

If $\gcd(m_0, n_0) \geq 2 \Rightarrow l = \gcd(m_0, n_0)$

$$X^{l m_1} - Y^{l n_1} = \underbrace{(X^{m_1} - Y^{n_1})}_{\text{Irred}} \underbrace{(X^{m_1(l-1)} + \cdots + Y^{n_1(l-1)})}_{\text{Contains another Irred.}}$$

$\Rightarrow (0,0)$ singular

$\therefore C \setminus \{0,0\}$: regular

$m = p^s m_0$, $n = p^t n_0$ $\quad s \geq t$

If $n_0 = 1$ or $s = t$, $m_0 = 1 \Rightarrow (0,0)$: regular

Otherwise $\Rightarrow (0,0)$: singular

Q3 Normality $S = V(XY - Z^2 =: F) \subset \mathbb{A}^3_k$. F: Irred $\frac{\partial F}{\partial X} = Y$, $\frac{\partial F}{\partial Y} = X$, $\frac{\partial F}{\partial Z} = -2Z$

$\Rightarrow Sing(S) = \{(0,0,0)\}$

Step 1 $k[X,Y,Z]/(XY - Z^2) \cong \underbrace{k[x^2, xy, y^2]}_{\text{sum of even degree monomials}} \subset k[x,y]$

Pf) Define

$$k[X,Y,Z] \xrightarrow{\varphi} k[x,y] \quad X \mapsto x^2,\ Y \mapsto y^2,\ Z \mapsto xy$$

$\Rightarrow \ker\varphi = \langle XY - Z^2 \rangle$ $\quad$ $a \in \ker\varphi$

($\supseteq$) Obvious

($\subseteq$) $a = (XY - Z^2)u + v$ $\quad$ $\deg_Z(v) \le 1$

$v = v_0 Z + v_1$. $\quad v_0, v_1 \in k[X,Y]$

$\varphi(a) = 0$ $\quad \underbrace{v_0(x^2, y^2)xy}_{\text{odd}} + \underbrace{v_1(x^2, y^2)}_{\text{even}} = 0$. $\quad \Rightarrow v_0 = v_1 = 0$

Step 2 $k[x^2, xy, y^2] = k[x,y] \cap k(x^2, xy, y^2) \subset k(x,y)$

($\subseteq$) Obvious

($\supseteq$) $h(x,y) = \frac{f(x^2, xy, y^2)}{g(x^2, xy, y^2)}$ $\quad f(x^2, xy, y^2) = g(x^2, xy, y^2) \cdot h(x,y)$

$h = h_0 + h_1 + \cdots + h_n$ $\quad$ homogeneous decomposition

$f = f_0 + f_2 + \cdots + f_{2s}$

$g = g_0 + g_2 + \cdots + g_{2t}$

$h_n = f_{2s} \cdot g_{2t} \Rightarrow n = \text{even}$

$f - f_n \in k[x,y] \cap k[x^2, xy, y^2] \Rightarrow f$ = some of even deg. hom. poly

$\Rightarrow f \in k[x^2, xy, y^2]$

Step 3 $k[x^2, xy, y^2]$: normal $\quad k[x^2, xy, y^2] \hookrightarrow k(x^2, xy, y^2) \ni \alpha$

$k[x,y]$: UFD $\Rightarrow$ normal

$\alpha^m + u_{m-1}\alpha^{m-1} + \cdots + u_m = 0$ $\quad u_i \in k[x^2, xy, y^2]$

$\alpha \in k(x,y)$ $\quad k[x,y]$: UFD $\Rightarrow \alpha \in k[x,y]$.

Q4) $X = V(f_1 \cdots f_r) \subset \mathbb{A}^n_k$

$$T(X) := \left\{ (x,v) \in X \times \mathbb{A}^n_k : \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(x) \cdot v_i = 0 \ , \ \forall j \right\}$$ affine ?

Consider

$$\mathbb{A}^{2n}_{(x_1 \cdots x_n, y_1 \cdots y_n)} \supset V\left(f_1 \cdots f_r, \ \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} \cdot y_i\right)$$

$\parallel \leftarrow$ as a set

$T(X)$

$\Rightarrow T(X)$ is an affine variety. $\boxed{}$