Exercise Sheet 1

Algebraic Geometry

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March 25, 2022

Exercise 1. *Let $F_1, \dots, F_r$ be homogeneous polynomials in $k[x_1, \dots, x_n]$ and let $X = Z(F_1, \dots, F_r)$ be the closed algebraic subset they define. Show that X is a cone with apex at the origin: that is, show that if x is a point in X, the line joining x to the origin lies entirely in X.*

Answer. Assume that $x \in X$. Then the line joining x and the origin is parametrized by $t \mapsto tx$ for $t \in k$. Notice that $tx \in X$ if and only if $F_i(tx) = 0$ for all $i \in \{1, \dots, r\}$. However, $F_i(tx) = t^d F_i(x) = 0$ where $d = deg(F_i)$ and we conclude.

Exercise 2. *Let $M_{n,m}$ be the space of $n \times m$-matrices with coefficients from k. It can be identified with the affine space $\mathbb{A}^{nm}$ with coordinates x_{ij} where $1 \leq i \leq n$ and $1 \leq j \leq m$. Let r be a natural number less than both n and m and let W_r be the set of $n \times m$-matrices of rank at most r. Show that W_r is a closed algebraic subset. Show that all the W_r's are cones over the origin. Hint: Determinants are polynomials.*

Answer. Notice that a matrix has rank at most r if and only if all its minors of rank $r+1$ are zero. Let $M = (X_{ij})_{ij} \in k[X_1, \dots, X_{nm}]$. Then, the minors $F_1, \dots, F_k$ of M of rank $r+1$ are homogeneous polynomials of degree $r+1$. Moreover, for any $A \in M_{n,m}$ the minors of A are exactly $F_i(A)$ (that is to say, the polynomial F_i evaluated at the entries in A). Consequently $W_r = Z(F_1, \dots, F_k)$and we conclude by exercise 1.

Exercise 3. *For every natural number $d \geq 2$ let $C_d \subseteq \mathbb{A}^d$ be the curve defined by the parametric representation $\phi(t) = (t, t^2, \ldots, t^n)$ and let $\mathfrak{a}$ be the ideal $\mathfrak{a} = (x_i - x_1 x_{i-1} : 2 \leq i \leq d)$. Show that C_d is a closed algebraic set, that $I(C_d) = \mathfrak{a}$ and $\mathfrak{a}$ is a prime ideal.*

Answer. First, observe that $C_d \subseteq Z(\mathfrak{a})$ is obvious since $t^i - t \cdot t^{i-1} = 0$. Conversely, let $(x_i)_i \in Z(\mathfrak{a})$. Inductively, $x_i = x_1^i$. Indeed, for $i = 1$, this is obvious and by $x_i = x_1 x_{i-1}$ we conclude that $(x_i)_i = \phi(x_1) \in C_d$. Consequently $I(C_d) = I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

It remains to show that $\mathfrak{a}$ is prime: Consider the k-algebra morphism $\psi : k[x_1, \ldots, x_n] \to k[x]$ defined by $\psi(x_i) = x^i$. Then $\psi(x_i - x_1 x_{i-1}) = x^i - x^i = 0$ and ψ induces a map $\overline{\psi} : k[x_1, \ldots, x_n]/\mathfrak{a} \to k[x]$. We claim that this is an isomorphism. To show this, we construct an inverse: Let $\eta : k[x] \to k[x_1, \ldots, x_n]/\mathfrak{a}$ be defined by $\eta(x) = [x_1]$. Then observe that

$$\overline{\psi} \circ \eta(x) = \overline{\psi}([x_1]) = x$$

and since x generates $k[x]$ as a k-algebra, $\overline{\psi} \circ \eta$ is the identity. Conversely, for every $i \in \{1, \ldots, n\}$:

$$\begin{aligned} \eta \circ \overline{\psi}([x_i]) &= \eta(x^i) = [x_1^i] \\ &= [x_1^{i-1} x_2] = \cdots = [x_i]. \end{aligned}$$

Analogously to the above, we conclude that $\eta \circ \overline{\psi}$ is the identity and $\overline{\psi}$ is an isomorphism. Now, since $k[x]$ is an integral domain $\mathfrak{a}$ is prime. $\square$