Exercise Sheet 2

Algebraic Geometry

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Exercise 1. *Show that any nonempty Hausdorff space X has dimension 0.*

Answer. First choose $x \in X$. Then $\{x\}$ is irreducible (and hence a chain of irreducible subspaces of length 0) and $dim(X) \geq 0$.

Conversely, let $Y \subseteq X$ be closed. Assume that $|Y| \geq 2$. Then there exists $x_1, x_2 \in Y$ with $x_1 \neq x_2$. But then, since Y is Hausdorff, there exist open neighbourhoods U and V of x_1 and x_2 with $U \cap V = \emptyset$. Equivalently $U^{\complement} \cup V^{\complement} = Y$ and since $x_1 \notin U^{\complement}$ and $x_2 \notin V^{\complement}$, the $U^{\complement}$ and $V^{\complement}$ are closed proper subspaces. Hence Y is not irreducible.

It finally follows that the non-empty irreducible subsets of X are precisely the singletons and in particular any chain of irreducible subsets has length at most 0, showing $dim(X) \leq 0$. □

Exercise 2. *Assume that $Y = Y_1 \cup \cdots \cup Y_r$ is the decomposition of the Noetherian space Y into irreducible components. Show that $dim(Y) = \max dim(Y_i)$ for Y_i irreducible components.*

Answer. Let $Z_0 \subsetneq \cdots \subsetneq Z_k$ be a chain of irreducible subsets of Y_i. Then, this is also a chain of irreducible subsets of Y and thus $k \leq dim(Y)$. Taking the supremum over k yields $dim(Y_i) \leq dim(Y)$. Then taking the maximum over all i shows

$$\max dim(Y_i) \leq dim(Y).$$

Conversely, let $Z_0 \subsetneq \cdots \subsetneq Z_k$ be a chain of irreducible subsets of Y. Consider

$i \in \{1, \dots, r\}$ such that $Z_k \cap Y_i \neq \emptyset$. Assume that $Z_k \not\subseteq Y_i$. Then notice that:

$$Z_k = (Z_k \cap Y_i) \cup \left(Z_k \cap \bigcup_{i \neq j} Y_j \right).$$

Since the Y_i are closed, this is a decomposition of Z_k into two proper closed subsets of Z_k contradicting its irreducibility. Therefore indeed $Z_k \subseteq Y_i$ and $k \leq dim(Y_i)$. Taking the maximum over the i and the supremum over k then shows the converse inequality:

$$\max dim(Y_i) \geq dim(Y).$$

□

Exercise 3. *Let $\psi : \mathbb{A}_k^2 \to \mathbb{A}_k^2$ be the map $\psi(x, y) = (x, xy)$. Determine the ideals $\psi^* \mathfrak{m}_{(a,b)}$ and the fibres $\psi^{-1}(a, b)$ for all points $(a, b) \in \mathbb{A}_k^2$.*

Answer. We have:

$$\begin{aligned} \psi^* \mathfrak{m}_{(a,b)} &= \psi^*((x-a, y-b)) \\ &= (\psi^*(x-a), \psi^*(y-b)) \\ &= (x-a, xy-b). \end{aligned}$$

Moreover, by case analysis:

$$\psi^{-1}(a,b) = \begin{cases} \{(a, b/a)\} & a \neq 0 \\ \{0\} \times \mathbb{A} & a = b = 0 \\ \emptyset & a = 0, b \neq 0. \end{cases}$$

Exercise 4. *Let X be a topological space. Show that the inclusion functor $Shv(X) \to PreShv(X)$ has a left adjoint.*

Answer. Denote by $I : Shv(X) \to PreShv(X)$ the inclusion functor. We must construct a functor $J : PreShv(X) \to Shv(X)$ such that for any $A \in PreShv(X)$, there exists a map $\epsilon_A : A \to I(J(A))$ satisfying the following universal property:

For any map of presheaves $\phi : A \to I(B)$, there exists a unique map of sheaves $\psi : J(A) \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} A & & \\ {\scriptstyle \epsilon_A}\downarrow & \overset{\phi}{\searrow} & \\ I(J(A)) & \xrightarrow[I(\psi)]{} & I(B) \end{array}$$

(see the definition of left/right adjoints via universal morphisms). Intuitively, $J(A)$ is the sheaf we get from A by changing A the least amount possible.

We construct $J(A)$ as follows: Observe that the only two additional properties we must ensure are locality and gluing. Hence a good guess would be two "add in" the missing glued morphisms and then quotient by an adequate equivalence relation. Therefore we define

$$K(A)(U) := \left\{ \{(f_i, U_i)\}_{i\in I} \;\middle|\; \begin{array}{l} U_i \subseteq X \text{ open, } \bigcup_{i\in I} U_i = U \\ f_i \in A(U_i) \text{ s.t. } f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j} \forall i,j \in I \end{array} \right\}.$$

Now, set

$$J(A)(U) := K(A)(U)/\sim$$

where $\sim$ captures locality: $\{(f_i, U_i)\} \sim \{(g_j, V_j)\}$ if and only if there exist $W_k \subseteq X$ open with $U = \bigcup_k W_k$ and for all i, j and k:

$$f_i|_{U_i\cap V_j\cap W_k} = g_j|_{U_i\cap V_j\cap W_k}.$$

First, we must show that this is indeed a sheaf. To this end, observe that the restriction defined by:

$$[\{(f_i, U_i)\}_{i\in I}]|_V = [\{(f_i|_{U_i\cap V}, U_i \cap V)\}_{i\in I}]$$

is well-defined and turns $J(A)$ into a presheaf. Moreover, locality is immediate from the definition of $\sim$.

Moreover, if $U \subseteq X$ is open, $U = \bigcup_{i\in I} U_i$ and $[\{(f_{i,j}, U_{i,j})\}_{j\in J_i}] \in J(A)(U_i)$ such that their restrictions agree on intersections, then:

$$[\{(f_{i,j}, U_{i,j})\}_{i\in I, j\in J_i}]$$

is an element of $J(A)(U)$ which restricts to the respective elements of $J(A)(U_i)$. Hence the gluing property holds and we have indeed constructed a sheaf.

Now, for the universal property, we first construct ϵ_A via

$$\epsilon_A(f) = [\{(f, U)\}]$$

for all $f \in A(U)$. Let B be an arbitrary sheaf and $\phi : A \to I(B)$ be a map of presheaves. Construct $\psi : J(A) \to B$ by defining $\psi([\{(f_i, U_i)\}_{i\in I}]) = g$ if and only if

$$g|_{\phi^{-1}(U_i)} = \phi(f_i|_{U_i})$$

for all i. Such a g always exists by the gluing property of B and ψ is well-defined by the definition of $\sim$ and the locality property of B. Observe that $\psi \circ \epsilon_A(f) = \psi([\{(f, U)\}]) = \phi(f)$ and hence the diagram commutes for this ψ. Moreover, any ψ which makes the diagram commute must be defined in this way, since ψ commutes with gluing and is therefore uniquely defined by its values on the elements $[\{(f, U)\}]$.

In conclusion, we have shown all of the properties stated above which guarantee the existence of a left-adjoint. □

Exercise 5. *Consider the real line $\mathbb{R}$ with the Euclidean topology.*

a. *Let $\mathcal{B}$ be the presheaf of bounded continuous real valued functions on $\mathbb{R}$. Show that $\mathcal{B}$ is not a sheaf.*
b. *What is the sheafification of $\mathcal{B}$?*

Answer. For part (a), consider the functions $f_n : (-n, n) \to \mathbb{R}$ defined by $f_n(x) = x$. Then the f_n are continuous and bounded, $f_n \in \mathcal{B}((-n, n))$. We claim that there does not exist an $f \in \mathcal{B}(\mathbb{R})$ such that $f|_{(-n,n)} = f_n$. Indeed, if this were the case, then we would have to have that $f(x) = x$ for all $x \in \mathbb{R}$. Since this is not a bounded function, we conclude.

For (b), we again consider the universal property of the sheafification $\overline{\mathcal{B}}$:

There exists a map $i : \mathcal{B} \to \overline{\mathcal{B}}$ such that for any sheaf A and $\phi : \mathcal{B} \to A$ a map of presheaves, there exists a unique map of sheaves $\psi : \overline{\mathcal{B}} \to A$ such that $\psi \circ i = \phi$.

We claim that $\overline{\mathcal{B}} = R_{\mathbb{R}}$ is the sheaf of continuous real-valued functions. Indeed, let $i : \mathcal{B} \to R_{\mathbb{R}}$ be the obvious inclusion. Moreover, let A be a sheaf and $\phi : \mathcal{B} \to A$ a map of presheaves. Then consider $f \in R_{\mathbb{R}}(U)$. Since f is continuous, it is locally bounded. We define $\psi(f)$ to be the unique element $g \in A(\phi^{-1}(U))$ such that $g|_{\phi^{-1}(V)} = \phi(f|_V)$ for every $V \subseteq U$ open such that $f|_V$ is bounded. This is well-defined by locality and gluing in A. Moreover, ψ is indeed a map of sheaves which satisfies $\psi \circ i = \phi$. Finally, ψ is unique, since every continuous function is obtained from bounded function by gluing and therefore ψ is uniquely defined by its values on the image of i. □