Exercise Sheet 3

Algebraic Geometry

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Exercise 1. *Let X be an algebraic variety over k and Z a closed subset of X. Consider the k-algebra sheaf defined as:*

$$\mathcal{O}_Z(U) = \left\{ f \text{ continuous on } U \,\middle|\, \begin{array}{l} f \text{ is locally the restriction of} \\ \text{regular functions on } X \end{array} \right\}.$$

Prove that $(Z, \mathcal{O}_Z)$ is an algebraic variety.

Answer. First, observe that Z is an algebraic variety if and only if it is locally isomorphic to an algebraic set. Hence it is sufficient by restricting to open subsets of X to consider the case when $(X, \mathcal{O}_X) \cong (F, \mathcal{O}_F)$ for F an algebraic set. Let $\phi : F \to X$ be the homeomorphism underlying this isomorphism of ringed spaces. Notice that $\phi^{-1}(Z) \subseteq F$ is closed and hence an algebraic set. We claim that Z is isomorphic to $\phi^{-1}(Z)$ as a ringed space. To see this, it suffices to show that ϕ induces a well-defined, bijective map between the regular functions on Z and the regular functions on $\phi^{-1}(Z)$.

To do so, we must more precisely show that for any $f : U \to k$ with $U \subseteq Z$ open, $\phi^* f$ is regular on $\phi^{-1}(U)$ if and only if f is regular on U. Equivalently, we show that f is regular at $x \in U$ if and only if $\phi^* f$ is regular at $\phi^{-1}(x)$: First, let f be regular at x. Then there exists $V \subseteq X$ open, $x \in V$ and $g \in \mathcal{O}_X(V)$ such that $f|_{V \cap Z} = g|_{V \cap Z}$. But then:

$$\begin{aligned} \phi^*(f)|_{V \cap Z} &= \phi^*(f|_{V \cap Z}) \\ &= \phi^*(g|_{V \cap Z}) \\ &= \phi^*(g)|_{V \cap Z}. \end{aligned}$$

Since being regular at x is a local property and $\phi^*(g)$ is regular at x, so is $\phi^*(f)$.

Conversely, let $\phi^*(f)$ be regular at $\phi^{-1}(x)$. Then there exists $V \subseteq \phi^{-1}(U)$ open with $\phi^{-1}(x) \in V$ and $\phi^*(f)|_V = \frac{g}{h}$ for $g, h \in k[\phi^{-1}(Z)]$. But now observe:

$$\begin{aligned} f|_{\phi(V)} &= (\phi^{-1})^*(\phi^*(f)|_V) \\ &= (\phi^{-1})^* \left(\frac{g}{h}\right)\Big|_{\phi(V)}. \end{aligned}$$

Since ϕ^{-1} is a map between k-ringed spaces, $(\phi^{-1})^*(g/h)$ is regular on X and hence f is the restriction of a regular function on X locally around x and we conclude the proof. □

Exercise 2. *Show that the embedding $j : \mathbb{A}^2 \setminus \{0\} \to \mathbb{A}^2$ induces an isomorphism of regular functions, i.e. sections of $\mathcal{O}_{\mathbb{A}^2}$.*

Answer. Notice that j induces a map of sheaves:

$$\overline{j} : (\mathbb{A}^2, \mathcal{O}_{\mathbb{A}^2}) \to (\mathbb{A}^2 \setminus \{0\}, \mathcal{O}_{\mathbb{A}^2 \setminus \{0\}})$$

given by restriction. We show that $\overline{j}$ is an isomorphism. For the injectivity, assume that $\overline{j}(g) = \overline{j}(f)$. Hence $g|_{\mathbb{A}^2 \setminus \{0\}} = f|_{\mathbb{A}^2 \setminus \{0\}}$. But now, $\mathbb{A}^2 \setminus \{0\}$ is an open subset of the irreducible set $\mathbb{A}^2$ and therefore a dense subset. In particular, if the continuous functions f and g agree on $\mathbb{A}^2 \setminus \{0\}$, they must agree on $\mathbb{A}^2$ and hence $f = g$.

For the surjectivity, let f be a regular function on $\mathbb{A}^2 \setminus \{0\}$. Then notice that f is in particular regular on $D(x)$ and hence by the characterization given in the lecture, $f = \frac{y^l g}{x^k}$ on $D(x)$ where $g \in k[x, y]$ is coprime to x and y and $k \geq 0$. Analogously $f = \frac{x^n h}{y^m}$ on $D(y)$. Therefore on $D(xy)$:

$$\frac{x^n h}{y^m} = \frac{y^l g}{x^k} \Leftrightarrow x^{k+n} h = y^{l+m} g.$$

Since this holds on a dense open subset of $\mathbb{A}^2$, it holds on the entirety of $\mathbb{A}^2$ and thus too in $k[x, y] = k[\mathbb{A}^2]$. But now, since h and g are coprime to x and y, it follows that $k + n = l + m = 0$ and indeed $f = g = h$ on $D(x) \cup D(y) = \mathbb{A}^2 \setminus \{0\}$. We conclude that $f = \overline{j}(g)$ and $\overline{j}$ is surjective. □

Exercise 3. *Let X and Y be algebraic varieties and $U \subseteq X$, $V \subseteq Y$ open subsets. Assume that there is an isomorphism of algebraic varieties $\phi : U \to V$ and define $W = X \sqcup Y/\sim$ where the equivalence relation identifies U and V via ϕ. Define $\mathcal{O}_W$ as the sheaf of functions whose pullbacks to both X and Y are regular. Show that $(W, \mathcal{O}_W)$ is an algebraic variety with X and Y as its open subvarieties.*

Answer. Notice that it is sufficient to show that there exists an open cover of W where each element of the cover is an algebraic variety. Indeed, the open subsets of W which are isomorphic to an algebraic set are then precisely the open subsets of the elements of the cover which satisfy this requirement.

Let $\pi : X \sqcup Y \to W$ be the canonical projection. We claim that $\pi(X) \cup \pi(Y) = W$ is such a cover. Indeed, note that $\pi^{-1}(\pi(X)) = X \cup V$ which is open in $X \sqcup Y$ and hence by symmetry this is indeed an open cover. It suffices again by symmetry to show that $\pi(X)$ is an algebraic variety. Now, notice that in fact $\pi|_{X \cup V} : X \cup V \to \pi(X)$ is an isomorphism of topological spaces. To see that it is in fact an isomorphism of ringed spaces we need only show that it induces a bijection on regular functions.

Note that indeed π induces a well-defined map on the set of regular functions by the definition of $\mathcal{O}_W$. Moreover, it is obviously surjective (π is in fact the pullback mentioned in the exercise statement when restricted to X and Y). Finally, injectivity follows from the fact that $\pi|_{X \cup V}$ is a bijective map (as map between topological spaces).

Consequently we need only show that $X \cup V$ is an algebraic variety. But now, observe that $X \sqcup Y$ is an algebraic variety (the open sets isomorphic to an algebraic set are inherited from X and Y by the properties of the disjoint union). Since $X \cup V$ is an open subset of an algebraic variety, the claim follows from the corresponding result shown in the lecture. $\square$