Exercise Sheet 4

Algebraic Geometry

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Exercise 1. *Let X be an irreducible algebraic set and let $k(X) = Frac(k[X])$ be its field of rational functions. Show that $k(X)$ can be defined as equivalence classes of regular functions defined on dense open subsets.*

Answer. Notice that since X is irreducible, any open subset is dense and we may therefore drop the assumption of denseness.

Now define $k\{X\}$ to be the set of regular functions defined on open subsets of X modulo the equivalence relation "$\sim$" where if $f : U \to k$ and $g : V \to k$ are regular on open sets, then $f \sim g$ if and only if $f|_{U\cap V} = g|_{U\cap V}$. Notice that $k\{X\}$ is a ring with ring structure induced by multiplication of regular functions since if f and g are as above, then fg is a regular function defined on $U \cap V$. We claim that

$$k[X] \cong k\{X\}.$$

Indeed, define $\phi : k[X] \to k\{X\}$ by

$$\phi\left(\frac{f}{g}\right) = \left[\frac{f}{g}\right].$$

Here, the right hand side is indeed an element of $k\{X\}$ since f/g is a regular function defined on $D(g)$. Moreover, ϕ is well-defined since if $\frac{f}{g} = \frac{q}{p}$ in $k(X)$, this is in particular true as functions from $D(g) \cap D(p)$ to k. Moreover, ϕ is obviously a ring morphism. Now:

$$\phi\left(\frac{f}{g}\right) = 0 \Leftrightarrow \frac{f}{g} = 0 \text{ on some non-empty open set } U.$$

In particular, $f = 0$ on U. Since $U \subseteq X$ is dense, $f = 0 \in k[X]$ and $\frac{f}{g} = 0$ showing that ϕ is injective.

For surjectivity, let $p : U \to X$ be a regular function defined on an open subset of X. Since the equivalence class $[p]$ is invariant under taking restrictions and the distinguished open subsets form a basis, we may assume $U = D(h)$ for some $h \in k[X]$. But then, from the lecture, we know that $p = \frac{g}{h^n}$ for some $g \in k[X]$ and $n \in \mathbb{N}$. Hence

$$[p] = \phi\left(\frac{g}{h^n}\right)$$

and ϕ is surjective. □

Exercise 2. *Show that any two different lines in $\mathbb{P}^2_k$ meet exactly in one point.*

Answer. We define lines to be the zero-sets of linear polynomials. Since we are considering the projective space, it is necessary to require that the linear polynomial be homogeneous in order to make this well-defined.

Hence consider $h_1(x, y, z) = a_1x + b_1y + c_1z \in k[x, y, z]$ and $h_2(x, y, z) = a_2x + b_2y + c_2z \in k[x, y, z]$ arbitrary with $a_i, b_i, c_i \in k \setminus \{0\}$. In order for $Z(h_1)$ and $Z(h_2)$ to be distinct lines, we must have that (a_1, b_1, c_1) and (a_2, b_2, c_2) are linearly independent. Namely because:

$$[z_1 : z_2 : z_3] \in Z(h_i) \Leftrightarrow (z_1, z_2, z_3) \perp (a_i, b_i, c_i).$$

But then:

$$[z_1 : z_2 : z_3] \in Z(h_1) \cap Z(h_2) \Leftrightarrow (z_1, z_2, z_3) \in \langle (a_1, b_1, c_1), (a_2, b_2, c_2) \rangle^{\perp}.$$

Since the orthogonal complement of a 2-dimensional subspace of k^3 is 1-dimensional and $[z_1 : z_2 : z_3] = [\lambda z_1 : \lambda z_2 : \lambda z_3]$ for any $\lambda \in k \setminus \{0\}$ it follows that $Z(h_1) \cap Z(h_2)$ contains exactly one element. □

Exercise 3. *Assume that $k = \overline{\mathbb{F}_p}$. Consider the Frobenius map $Frob : \mathbb{A}^n_k \to \mathbb{A}^n_k$ that sends $(a_0, \ldots, a_n)$ to $(a_0^p, \ldots, a_n^p)$. Show that it is a bijection but not an isomorphism. Explain why fixed points of $Frob^m$ are the points with coordinates in $\mathbb{F}_m$.*

Answer. For the proof we use the following lemma from general algebra:

Lemma 0.1.

$$k := \overline{\mathbb{F}_p} = \bigcup_{m=1}^{\infty} \mathbb{F}_{p^m}$$

where $\mathbb{F}_{p^m}$ is a subfield and $\mathbb{F}_{p^m}^\times$ is the zero-set of the polynomial $X^{p^m-1}-1$.

Proof. Notice that every element of $k^\times$ has finite order in $k^\times$. Indeed, let $x \in k^\times$. Then x is algebraic over $\mathbb{F}_p$. Therefore $\mathbb{F}_p[x]$ is a finite dimensional vector space over $\mathbb{F}_p$ and in particular finite. Moreover, $\mathbb{F}_p[x] = \mathbb{F}_p(x)$ (this is a general fact about algebraic elements over a field) and hence x must have finite order since it is an element of the finite group $\mathbb{F}_p(x)^\times$.

Finally, notice that $|\mathbb{F}_p(x)^\times| = p^m - 1$ where m is the dimension of $\mathbb{F}_p(x)$ over $\mathbb{F}_p$. Therefore by Lagrange every element of $\mathbb{F}_p(x)^\times$ is a zero of the polynomial $X^{p^m-1}-1$. Since this polynomial has at most $p^m - 1$ zeros, we conclude that $\mathbb{F}_p(x) = \mathbb{F}_{p^m}$.

To see that for each m, $\mathbb{F}_{p^m}$ is indeed a subfield, observe that for all $x, y \in \mathbb{F}_{p^m}^\times$: $x^{-1} = x^{p^m-2} \in \mathbb{F}_{p^m}$, $x \cdot y \in \mathbb{F}_{p^m}$ and $(x+y)^{p^m} = x^{p^m} + y^{p^m} = x + y$, hence $x + y \in \mathbb{F}_{p^m}$. □

Now we return to the exercise: Notice that Frob restricts to every $\mathbb{F}_{p^m}$ and induces the Frobenius automorphism on $\mathbb{F}_{p^m}$ (since $x^{p^m} = x$ for all $x \in \mathbb{F}_{p^m}$, $\text{Frob}^{-1} = \text{Frob}^{m-1}$). Hence Frob maps $\mathbb{F}_{p^m}$ bijectively onto $\mathbb{F}_{p^m}$ and it follows that it is a bijection on k.

It remains to show that Frob is not an isomorphism. To see this, we show that Frob is not surjective on regular functions. More precisely, we show that $\text{Frob}^* : \mathcal{O}_k(\mathbb{A}_k^n) \to \mathcal{O}_k(\mathbb{A}_k^n)$ is not surjective. From the lecture, we know that

$$\mathcal{O}_k(\mathbb{A}_k^n) \cong k[\mathbb{A}_k^n] = k[x_1, \ldots, x_n].$$

Now, observe that for $f \in \mathcal{O}_k(\mathbb{A}_k^n)$:

$$\text{Frob}^*(f)(a_1, \ldots, a_n) = f(a_1^p, \ldots, a_n^p).$$

Hence under the above isomorphism, Frob^* is the map

$$\phi := ev_{x_1^p, \ldots, x_n^p} : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n].$$

However, this map is not surjective. Indeed, observe that $p | deg_{x_1}(\phi(f))$ for any $f \in k[x_1, \ldots, x_n]$. Consequently for example $x_1 \notin im(\phi)$ and we conclude.

Finally, for the statement about the fixed points of Frob^m, notice that a point in $\mathbb{A}_k^n$ is fixed if and only if every coordinate is fixed. Now, an element $x \in k$ is fixed by Frob^m if and only if $x^{p^m} - x = 0$. Since $x^{p^m} - x = x(x^{p^m-1} - 1)$, the fixed points of the Frobenius homomorphism are exactly the elements of $\mathbb{F}_{p^m}$. □