Exercise Sheet 6

Algebraic Geometry

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Exercise 1. *Let $k = \mathbb{C}$. Show that there exists a continuous map (in Zariski topology)*

$$f : \mathbb{A}_k \to \mathbb{A}_k$$

which is not a regular function.

Answer. Define $f : \mathbb{A}_k \to \mathbb{A}_k$ by $f(x_1, \ldots, x_k) = (\overline{x_1}, \ldots, \overline{x_k})$. This function is continuous. Indeed, let $g \in k[x_1, \ldots, x_k]$. Then $f^{-1}(Z(g)) = Z(\overline{g})$ where $\overline{g}$ is the polynomial whose coefficients are the complex conjugates of the coefficients of g. Since this set is closed, f is continuous.

We claim that f is not regular. Indeed, if f were regular, then as shown in the lecture, it must hold that f is a polynomial. Hence so too is:

$$h := f \cdot (x_1, \ldots, x_k) - i.$$

But now observe that $f(x_1, \ldots, x_k) \cdot (x_1, \ldots, x_k) \in \mathbb{R}$ for all $(x_1, \ldots, x_k) \in \mathbb{C}^k$. Consequently h has no zeros and is hence constant. However, this is only possible if $f = 0$ which is not the case. This contradiction shows that f is not a polynomial. $\square$

Exercise 2. *In the class we showed that for affine varieties X and Y, $dim(X \times Y) = dim(X) + dim(Y)$. Deduce this formula for general algebraic varieties X and Y (possibly non-irreducible).*

Answer. Let X and Y be as in the exercise. Write

$$X = \bigcup_{i=1}^{n} U_i \qquad Y = \bigcup_{j=1}^{m} V_j$$

where U_i and V_j are affine open subvarieties. Notice that

$$X \times Y = \bigcup_{i,j} U_i \times V_j$$

as sets. We equip the right-hand space with the topology induced by the Zariski topology on $U_i \times V_j$ and claim that the equality above is an equality of topological spaces and in fact describes an open cover of $X \times Y$ by open subsets. Showing this claim will constitute the main part of the exercise.

Indeed, notice that there are canonical, continuous projections:

$$p_X := \bigcup_{i,j} U_i \times V_j \to X \quad p_Y := \bigcup_{i,j} U_i \times V_j \to Y.$$

Since these are regular maps, by the universal property of the product, there exists a unique regular function

$$\psi : \bigcup_{i,j} U_i \times V_j \to X \times Y$$

such that $\pi_X \circ \psi = p_X$ and $\pi_Y \circ \psi = p_Y$, where π_X and π_Y are the canonical projections of $X \times Y$. In particular, for all $(x, y) \in \bigcup_{i,j} U_i \times V_j$:

$$\psi(x, y) = (\pi_X \circ \psi(x), \pi_Y \circ \psi(y)) = (x, y)$$

and thus ψ is the identity as a set of maps. This shows that any set which is open in $X \times Y$ is also open in $\bigcup_{i,j} U_i \times V_j$. Conversely, observe that

$$U_i \times V_j = \pi_X^{-1}(U_i) \cap \pi_Y^{-1}(V_j)$$

is open in $X \times Y$. Again, by the universal property of the product, there exists a unique regular function $U_i \times V_j \subseteq X \times Y \to U_i \times V_j$. This is the restriction of the identity and in particular, any open subset of $U_i \times V_j$ is open in $X \times Y$.

This shows the claim.

Now, as shown in the lecture:

$$dim(X \times Y) = \max_{i,j} dim(U_i \times V_j).$$

But $dim(U_i \times V_j) = dim(U_i) + dim(V_j)$ by the statement mentioned in the exercise and thus

$$dim(X \times Y) = \max_{i} dim(U_i) + \max_{j} dim(V_j) = dim(X) + dim(Y).$$

□

Exercise 3. *Let $f : X \to Y$ be a finite morphism. Show that f has finite fibres.*

Answer. First, since any finite morphism is affine by definition, we may assume that X and Y are affine. Let $y \in Y$. We aim to show that $f^{-1}(\{y\})$ is finite.

To see this, let $\mathfrak{m}_y \in k[Y]$ be the maximal ideal corresponding to y. Set

$$\mathfrak{a} := \sqrt{k[X]f^*(\mathfrak{m}_y)} = \sqrt{(f^*(x_i - y_i) : i \in \{1, \dots, n\})}$$

where $Y \subseteq \mathbb{A}^n$. Notice that $g \in \mathfrak{a}$ if and only if g is zero everywhere where $f^*(x_i - y_i)$ is zero which is the case if and only if $g \in I(f^{-1}(\{y\}))$. Consequently $\mathfrak{a} = I(f^{-1}(\{y\}))$. Moreover, observe that f^* induces a morphism of rings:

$$\phi : k[Y]/\mathfrak{m}_y \to k[X]/\mathfrak{a}.$$

Since f is finite, ϕ is integral. Hence, since $k[Y]/\mathfrak{m}_y$ is a field (which also implies that ϕ is injective), by "going-up", $dim(k[X]/\mathfrak{a}) = 0$. Also, since $k[X]$ is Noetherian, so is the quotient and by Akizuki, it is Artinian. In particular, it has finitely many maximal ideals. But now, by the equality $\mathfrak{a} = I(f^{-1}(\{y\}))$ shown above, $k[X]/\mathfrak{a} = k[f^{-1}(\{y\})]$. Hence the maximal ideals correspond bijectively to points in $f^{-1}(\{y\})$ and we conclude that this set is finite. □

Exercise 4. *Let A be the ring defined by $k[x]_{(x)}[t]$. Show that there exists a codimension one prime ideal $\mathfrak{p} \subseteq A$ which has height zero. Note that $dim(A) = 2$.*

Answer. Notice that for $\mathfrak{p}$ to have dimension 0 (which is the dimension of the quotient by $\mathfrak{p}$), we must have that $\mathfrak{p}$ is maximal. For example, define $\mathfrak{p} := (xt-1)$ (hence t is intuitively the "missing" inverse of x). Define $\phi : k[x]_{(x)}[t] \to k(x)$ to be the map induced by the evaluation $(x, t) \mapsto (x, x^{-1})$. Obviously, ϕ is surjective. Moreover, $\phi(\mathfrak{p}) = (0)$. Conversely, assume $\phi(a) = 0$. By clearing denominators, we may assume $a \in k[x][t]$. But then, this implies that x^{-1} is a zero of the polynomial a viewed as a polynomial in t with coefficients in $k(x)$. Hence $xt - 1|a$ in $k(x)[t]$. By Gauss's lemma, since $xt - 1$ is primitive, $xt - 1|a$ in $k[x][t]$ and thus $a \in \mathfrak{p}$. We deduce that ϕ induces an isomorphism $k[x]_{(x)}[t]/\mathfrak{p} \to k(x)$ and since the latter is a field, $\mathfrak{p}$ is maximal.

Note that this isomorphism is in fact the reason for which we consider $(xt - 1)$ instead of the more intuitive maximal ideal (x, t). Indeed, the quotient by the latter is isomorphic to k which is much "smaller" than $k(x)$ and hence intuitively, the codimension of (x, t) must be larger.

It remains to show that $\mathfrak{p}$ has codimension (that is to say, height) one. However, this is indeed the case by Krull's principal ideal theorem ($\mathfrak{p}$ is the minimal prime containing $xt - 1$ which is not a zero-divisor). $\square$