Exercise Sheet 7

Algebraic Geometry

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Exercise 1. *Let X be an algebraic variety, p a point in X. Show that the differential map*

$$d_p : \mathcal{O}_{X,p} \to \mathfrak{m}_p/\mathfrak{m}_p^2$$

that sends a function f to the equivalence class of $f - f(p)$ satisfies the Leibniz rule.

Answer. Let $f, g \in \mathcal{O}_{X,p}$. Then:

$$\begin{aligned} f(p)d_p(g) + g(p)d_p(f) &= f(p)(g - g(p)) + g(p)(f - f(p)) + \mathfrak{m}_p^2 \\ &= f(p)g - f(p)g(p) + g(p)f - g(p)f(p) + \mathfrak{m}_p^2 \\ &= fg - g(p)f(p) + (f(p)g - f(p)g(p) + g(p)f - fg) + \mathfrak{m}_p^2 \\ &= d_p(fg) + (\phi + \mathfrak{m}_p^2) \end{aligned}$$

where

$$\phi = f(p)g - f(p)g(p) + g(p)f - fg.$$

Notice that

$$\phi = -(f - f(p))(g - g(p)) \in \mathfrak{m}_p^2$$

and thus we conclude that

$$f(p)d_p(g) + g(p)d_p(f) = d_p(fg).$$

□

Exercise 2. *Compute the singularities of the curve*

$$Z = Z(X^m - Y^n) \subseteq \mathbb{A}_k^2$$

over fields of arbitrary characteristics.

Answer. Notice that Z is defined by a single equation. Hence since the height of any minimal prime containing $X^m - Y^n$ is 1 by Krull's principal ideal theorem, the codimension of any irreducible component of Z is 1.

Now, for the dimension of the tangent space, let $p = (x, y) \in Z$. We first consider the ideal:

$$\mathfrak{m}_p = (X - x, Y - y) \subseteq k[X, Y]/(X^m - Y^n).$$

Notice that $X^m - Y^n$ need not be irreducible and hence $(X^m - Y^n)$ is not necessarily $I(Z)$. However, it is always a subset of $I(Z)$ and since quotients of vector spaces have lower dimension, calculating $\mathfrak{m}_p/\mathfrak{m}_p^2$ gives us an upper bound on the dimension of the tangent space. Since at any point, the dimension of the tangent space is larger or equal to the codimension, this gives us a sufficient condition for a point to be regular.

Now:

$$\begin{aligned}\mathfrak{m}_p/\mathfrak{m}_p^2 &\cong \frac{((X - x, Y - y) + (X^m + Y^n))}{(((X - x)^2, (Y - y)^2, (X - x)(Y - y)) + (X^m - Y^n))} \\ &\cong \frac{(X - x, Y - y)}{((X - x)^2, (Y - y)^2, (X - x)(Y - y), X^m - Y^n)}.\end{aligned}$$

Defining $a := X - x$ and $b := Y - y$:

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \cong (a, b)/(a^2, b^2, (a + x)^m - (b + y)^n).$$

But now, observe that

$$(a + x)^m - (b + y)^n = x^m + amx^{m-1} - y^n - bny^{n-1} + g$$

for some $g \in (a^2, b^2)$. Consequently since $x^m - y^n = 0$:

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \cong (a, b)/(a^2, b^2, amx^{m-1} - bny^{n-1}).$$

If mx^{m-1} or ny^{n-1} is not equal to zero (without loss of generality, assume $mx^{m-1} \neq 0$):

$$\begin{aligned}\mathfrak{m}_p/\mathfrak{m}_p^2 &\cong (a, b)/(a^2, b^2, a - (mx^{m-1})^{-1}bny^{n-1}) \\ &\cong (b)/(b^2) \cong kb\end{aligned}$$

which is a vector space of dimension 1. Hence, by the argument given above, whenever $mx^{m-1} \neq 0$ or $ny^{n-1} \neq 0$, (x, y) is a regular point. It remains to analyze the case when

$$mx^{m-1} = ny^{n-1} = 0. \tag{$*$}$$

In this case, the above vector space is isomorphic to $kX \oplus kY$ which is a vector space of dimension 2.

First if $m = 1$ or $n = 1$, then $(*)$ never holds. Consequently, we may assume that $m, n \geq 2$. Now, assume that $m = m'p^k$ and $n = n'p^l$ are both divisible by the characteristic p of k. Without loss of generality, $k \geq l$. Then notice:

$$X^m - Y^n = (X^{m'p^{k-l}} - Y^{n'})^{p^l}$$

and hence replacing (m, n) with $(m'p^{k-l}, n')$ we may assume that n and p are coprime. In this case, the only possibility for $(*)$ to hold is $y = 0$ and thus too $x = 0$. If $X^m - Y^n$ is irreducible, then the above shows that $(0, 0)$ is a singular point. Therefore, it remains to consider the cases in which n and p are coprime and $X^m - Y^n$ is reducible in $k[X, Y]$.

First, assume that m and n are not coprime. Consequently there exists $q > 1$ such that $m = qm'$ and $n = qn'$. It follows that:

$$\begin{aligned} X^m - Y^n &= (X^{m'})^q - (Y^{n'})^q \\ &= (X^{m'} - Y^{n'})(X^{m'(q-1)} + X^{m'(q-2)}Y^{n'} \cdots + Y^{n'(q-1)}). \end{aligned}$$

Since $(0, 0)$ is a zero of both of these factors, $(0, 0)$ is contained in at least two irreducible components of Z and as shown in the lecture, it is singular.

We remain with the case that m and n are coprime: By Gauss's lemma, it suffices to check whether $f := X^m - Y^n$ is irreducible in $k(X)[Y]$. We first instead notice that f is irreducible in $k[X^m, Y]$ by the same degree considerations as for the case $m = 1$. In particular, by Gauss's lemma, f is irreducible in $k(X^m)[Y]$. Observe that $k(X)$ is a finite extension of $k(X^m)$ of degree m. Since $deg_Y(f) = n$ is coprime to the degree of the extension, we conclude that f is also irreducible in $k(X)[Y]$.

Note: We can see that f remains irreducible in $k(X)$ as follows: Let $L|k(X)$ be a splitting field of f. Consider a root $\alpha \in L$ of f. Then by the multiplicativity of degrees of field extensions:

$$\begin{aligned} [k(X, \alpha) : k(X)][k(X) : k(X^m)] &= [k(X, \alpha) : k(X^m)] \\ &= [k(X, \alpha) : k(\alpha, X^m)][k(\alpha, X^m) : k(X^m)]. \end{aligned}$$

Now the left-hand side is equal to $[k(X, \alpha) : k(X)] \cdot m$. Moreover, the left-hand side is $[k(X, \alpha) : k(\alpha, X^m)] \cdot n$. Since m and n are coprime, it follows that $n | [k(X, \alpha) : k(X)]$ and some irreducible factor of f must have degree at least n. Since n is the degree of f, we conclude.

To summarize, we have found the following characterization:

Let $m = m'p^k$ and $n = n'p^l$ with m' and n' coprime to p where p is the characteristic of k (if p is 0, $m = m'$ and $n = n'$). Without loss of generality, $k \geq l$.

- Every point in $Z \setminus \{(0,0)\}$ is regular.
- $m'p^{k-l} = 1$ or $n' = 1$:
  - $\{(0,0)\}$ is a regular point.
- $m'p^{k-l}, n' \geq 2$:
  - $\{(0,0)\}$ is not regular.

Exercise 3. *Show that the quadric cone*

$$W = Z(XY - Z^2) \subseteq \mathbb{A}^3_k$$

is singular and normal (i.e., its coordinate ring is integrally closed).

Answer. Let $f = (XY - Z^2)$. Notice that $deg_X(f) = 1$ and hence $f \in k[X,Y,Z] = k[Y,Z][X]$ is irreducible. Now:

$$\partial_X f = Y$$
$$\partial_Y f = X$$
$$\partial_Z f = -2Z.$$

Thus by the Jacobian criterion, W has a singularity at $(0,0,0)$ showing that W is singular.

For the normality of W, notice that we have

$$k[W] = k[X,Y,Z]/(f) \cong k[s^2, t^2, st] \subseteq k[s,t]$$

via the map $\overline{\phi}$ induced by $\phi := ev_{(s^2,t^2,st)}$. Indeed, surjectivity of $\overline{\phi}$ is obvious and if $\phi(g) = 0$,then for every $x \in \mathbb{A}^3$ with $f(x) = 0$, we have

$$g(x) = g(\sqrt{x_1}^2, \sqrt{x_2}^2, \sqrt{x_1 x_2}) = \phi(g)(\sqrt{x_1}, \sqrt{x_2}) = 0$$

which shows by the Nullstellensatz that $g^n \in (f)$ for some $n \geq 0$. But (f) is prime since f is irreducible and we conclude that $g \in (f)$. This shows that $\overline{\phi}$ is injective. We must hence show that $k[s^2, t^2, st]$ is integrally closed. First notice that $k[s,t]$ is a UFD and hence integrally closed. Since $k[s^2, t^2, st] \subseteq k[s,t]$ this implies that any integral element of $Frac(k[s^2, t^2, st])$ must be contained in $k[s,t]$. Assume that $f \in k[s,t]$ is integral over $k[s^2, t^2, st]$ and not contained in the latter ring. Write $f = \frac{g}{h}$ where $g, h \in k[s^2, t^2, st]$. Then we obtain an equation $k[s,t]$:

$$fh = g.$$

Since by assumption $f \notin k[s^2, t^2, st]$, there exists some monomial of f which is not a product of s^2, t^2 and st. For an arbitrary monomial $s^\alpha t^\beta$, this is the case if and only if α and β have different parity. But now, order the monomials of f first by their degree in t and then by their degree in s. Consider the (unique) maximal monomial with respect to this order in f which is not contained in $k[s^2, t^2, st]$. Multiplying this monomial with the maximal monomial in h yields a unique maximal monomial in fh whose degree in t and s have different parity (since all monomials in h are contained in $k[s^2, t^2, st]$). Since $fh = g \in k[s^2, t^2, st]$, this is a contradiction and we conclude that no such f exists. In particular, $k[s^2, t^2, st]$ is integrally closed. $\square$

Exercise 4. *For*

$$X = Z(f_1, \ldots, f_r) \subseteq \mathbb{A}_k^n$$

with f_i prime, we define the tangent bundle of X as the set

$$T(x) := \left\{ (x, v) \in X \times \mathbb{A}_k^n : \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(x) \cdot v_i = 0, \, \forall j \right\}.$$

Show that $T(X)$ is an affine variety.

Answer. We consider the polynomials f_i as polynomials in $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Moreover, define

$$g_j := \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(x) y_i \in k[y_1, \ldots, y_n] \subseteq k[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

Then

$$T(X) = Z(f_1, \ldots, f_r, g_1, \ldots, g_r) \subseteq \mathbb{A}_k^{2n}$$

and we conclude.