Exercise Sheet 8

Algebraic Geometry

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Exercise 1. *Let $f, g : X \to Y$ be morphisms between algebraic varieties. Suppose Y is separated. Prove the following statements*

a. The graph $\Gamma_f = \{(x, f(x)) \in X \times Y\}$ is closed in $X \times Y$.
b. The set $\{x \in X : f(x) = g(x)\}$ is closed in X.
c. We further assume that X and Y are irreducible. Suppose there is a nonempty open subset $U \subseteq X$ such that $f|_U = g|_U$. Then $f = g$ on X.

Answer. a.

Let $\pi_X : X \times Y \to X$ be the canonical projection and analogously π_Y. Moreover, consider $\phi : X \times Y \to Y \times Y$ defined by $\phi(x, y) = (f(x), y)$ (this is the unique morphism induced by the morphisms $X \times Y \to Y, (x, y) \mapsto f(\pi_X(x, y))$ and $X \times Y \to Y, (x, y) \mapsto \pi_Y(x, y)$ via the universal property of the product). Observe that

$$\Gamma_f = \phi^{-1}(\{(y, y) : y \in Y\}).$$

Since this set is closed by separatedness of Y and continuity of ϕ, we conclude.

b.

Let $\phi : X \to Y \times Y$ be the morphism induced by the morphisms f and g via the universal properties of the product. Explicitly, $\phi(x) = (f(x), g(x))$. Then observe that

$$\{x \in X : f(x) = g(x)\} = \phi^{-1}(\{(y, y) : y \in Y\}).$$

As in (a), we conclude by separatedness of Y and continuity of ϕ.

c.

Set $A := \{x \in X : f(x) = g(x)\}$. By (b), A is closed. Moreover, by the condition given in the exercise, $U \subseteq A$. Hence $\overline{U} \subseteq A$. But since X is irreducible and U is a non-empty open set, U is dense and it follows that $A = X$ showing that $f = g$ on X. $\square$

Exercise 2. *An algebraic variety X is called rational if it is birationally equivalent to $\mathbb{P}^n$ for some n.*

a. Show that any conic in $\mathbb{P}^2$ is rational.
b. Show that the cuspidal cubic $Z(Y^2 - X^3) \subseteq \mathbb{A}^2$ is rational.

Answer. a.

Let $C \subseteq \mathbb{P}^2$ be a conic. That is to say, $C = Z_+(f)$ where $f \in k[x, y, z]$ is an irreducible homogeneous polynomial of degree 2. Let $U \subseteq \mathbb{P}^2$ be an affine open subset $U = D_+(x_k)$ of $\mathbb{P}^2$ such that $C \cap U \neq \varnothing$. Then $C \cap U$ is a dense open subset of C and it suffices to show that $C \cap U$ is rational. Pulling back f, we must hence show that any conic in $\mathbb{A}^2$ is rational. From the lecture, we know that up to isomorphism, there are exactly two such f:

- $f_1 = X^2 - Y$
- $f_2 = XY - 1$.

For f_1, consider $\phi : Z(f_1) \to \mathbb{A}^1$ defined by $\phi(X, Y) = X$ and $\psi : \mathbb{A}^1 \to Z(f_1)$ defined by $\psi(X) = (X, X^2)$. Then ϕ and ψ are inverse morphisms and hence $Z(f_1) \cong \mathbb{A}^1$. Since $\mathbb{A}^1$ is isomorphic to an open subset of $\mathbb{P}^1$, we conclude that $Z(f_1)$ is birationally equivalent to $\mathbb{P}^1$.

For f_2 consider instead $\phi : Z(f_2) \to \mathbb{A}^1 \setminus \{0\}$ defined by $\phi(X, Y) = X$ and $\psi : \mathbb{A}^1 \setminus \{0\} \to Z(f_2)$ defined by $\psi(X) = (X, X^{-1})$. These are regular functions and in fact are inverses to one another. Hence $Z(f_2)$ is birational to $\mathbb{A}^1$ and as above, we conclude.

b.

Let $f := Y^2 - X^3$, $Z := Z(f)$ and define $\phi : Z \dashrightarrow \mathbb{P}^1$ by

$$\phi(x, y) = (x : y).$$

This is defined on $Z \setminus \{(0, 0)\}$. Moreover, define $\psi : \mathbb{P}^1 \dashrightarrow Z$ by

$$\psi(x : y) = \left(\left(\frac{y}{x}\right)^2, \left(\frac{y}{x}\right)^3 \right).$$

This is defined on $D_+(x)$. We restrict ψ to $D_+(xy)$ such that the image of ψ lies within the domain of ϕ. Note also that the image of ϕ lies within $D_+(xy)$. Moreover, for all $(x, y) \in Z$:

$$\psi(\phi(x, y)) = (x, y).$$

This can be seen by using the relation $y^2 = x^3$. Conversely for all $(x:y) \in D_+(xy)$:

$$\phi(\psi(x:y)) = \left(\left(\frac{y}{x}\right)^2 : \left(\frac{y}{x}\right)^3\right) = (x:y)$$

after multiplying with $\frac{x^3}{y^2}$. This shows that $\psi|_{D_+(xy)}$ is an isomorphism and Z is birational to $\mathbb{P}^1$ and in particular rational. □

Exercise 3. *Let X and Y be two algebraic varieties. Suppose there are points $P \in X$ and $Q \in Y$ such that the local rings $\mathcal{O}_{X,P}$ and $\mathcal{O}_{Y,Q}$ are isomorphic k-algebras. Then show that there are open sets $P \in U \subseteq X$ and $Q \in V \subseteq Y$ and an isomorphism of U to V which sends P to Q.*

Answer. Since all conditions and claims are local, we may assume that X and Y are affine. Hence $\mathcal{O}_{X,P} = k[X]_{\mathfrak{m}_P}$ where $\mathfrak{m}_P$ is the maximal ideal corresponding to P. Analogously for Y. Consequently, the requirement in the exercise translates to:

$$k[X]_{\mathfrak{m}_P} \cong k[Y]_{\mathfrak{m}_Q}.$$

Let ϕ be such an isomorphism. Choose $f_1, \dots, f_n \in k[Y]_{\mathfrak{m}_Q}$ (where $X \subseteq \mathbb{A}^n$) such that

$$\phi\left(\frac{g}{h}\right) = \frac{g(f_1, \dots, f_n)}{h(f_1, \dots, f_n)}$$

for all $g \in k[X]$ and $h \in k[X] \setminus \mathfrak{m}_P$. After affine transformation we may assume $P = Q = 0$. Let $a \in \mathbb{A} \setminus \{0\}$ and consider the special case $g = 1$, $h = x_k - a$ where $k \in \{1, \dots, n\}$. Notice that for $\phi(g/h)$ to be an element of $k[Y]_{\mathfrak{m}_Q}$, we must have $(h(f_1, \dots, f_n))(0) = f_k(0) - a \neq 0$ (observe that $f_k(0) = \phi(x_k)(0)$ must be defined). Consequently $f_k(0) \neq a$ for any $a \in \mathbb{A} \setminus \{0\}$. We conclude that $f_k(0) = 0$. Let $V \subseteq Y$ be a neighbourhood of 0 such that each f_k is regular on V. V is an open subvariety of Y and restricting Y further, we may assume that the f_k are regular on Y (note that we must also choose an affine subvariety of V). But then each f_k is a polynomial. Hence ϕ restricts to a map $\phi : k[X] \to k[Y]$. Analogously, by further restricting X, the same procedure shows that ϕ^{-1} restricts to a map $\phi^{-1} : k[Y] \to k[X]$. Hence ϕ is an isomorphism of coordinate rings and by the main theorem of affine varieties, we conclude that ϕ induces an isomorphism between X and Y. Now, since each f_k in the above maps 0 to 0, so too does the induced isomorphism (alternatively, note that ϕ and ϕ^{-1} must map $\mathfrak{m}_P$ and $\mathfrak{m}_Q$ into one another) and we conclude. □

Exercise 4. *Prove that any two one dimensional irreducible algebraic varieties over k are homeomorphic. (Later we will see that there are non-birational curves)*

Answer. Notice that the closed subsets of $\mathbb{A}^1$ are precisely the finite sets. That being said, every one dimensional algebraic variety X over k is a finite union of sets homeomorphic to a cofinite subset of $\mathbb{A}^1$. In particular, X has the same cardinality as $\mathbb{A}^1$ and there exists a bijection (of sets) $\phi : X \to \mathbb{A}^1$. But now since ϕ is a bijection, the preimage of any finite subset of $\mathbb{A}^1$ is a finite subset of X showing that in fact ϕ is continuous. The same argument shows that ϕ^{-1} is continuous and ϕ is a homeomorphism. □