Exercise Sheet 9

Algebraic Geometry

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Exercise 1. *Describe $Bl_p\mathbb{P}^2$ as pairs (q,l) where l is a line in $\mathbb{P}^2$ through q and p.*

Answer. As defined in the lecture (page 12 of chapter 7):

$$Bl_p(\mathbb{P}^2) = Z_+(x_0y_1 - x_1y_0)$$

where $p = (0:0:1)$. Let $a := (q_a, (a_0 : a_1)) \in Z(XV - YU)$. We define $\phi(a) := (q_a, l_a)$ where l is the line which contains all $x \in \mathbb{P}^2$ such that $(a_1, -a_0, 0) \cdot x = 0$ (here, "$\cdot$" is the standard scalar product on k^3). This is indeed well-defined and in fact $(a_1, -a_0, 0) \cdot p = (a_1, -a_0, 0) \cdot q = 0$. We claim that ϕ is a bijection. For injectivity note that $\phi(a) = \phi(b)$ implies in particular $q_a = q_b$. Moreover, $l_a = l_b$ holds if and only if $(a_1, -a_0, 0)$ and $(b_1, -b_0, 0)$ are colinear which is the case if and only if $(a_0 : a_1) = (b_0 : b_1)$.

For surjectivity, let (q,l) be a pair as in the exercise. Then l is given by a homogeneous linear equation and we may hence write

$$x \in l \Leftrightarrow (y_0, y_1, y_2) \cdot x = 0$$

for some $(y_0 : y_1 : y_2) \in \mathbb{P}^2$. But now, since $p \in l$, we must have $0 = (y_0, y_1, y_2) \cdot p = y_2$. Moreover, writing $q = (q_0 : q_1 : q_2)$ it holds that $q \in l$ and consequently

$$q_0y_0 + q_1y_1 = 0.$$

Therefore $(q, (-y_1 : y_0)) \in Z(XV - YU)$ and in addition ϕ maps this element to (q,l). □

Exercise 2. *Let X be a topological space. Suppose that X has an open cover $\{U_i\}_{i\in I}$ where U_i are irreducible and $U_i \cap U_j \neq \varnothing$ for all $i, j \in I$. Then prove that X is irreducible.*

Answer. Recall that a topological space is irreducible if and only if every non-empty open subset of X is dense in X.

Let $U \subseteq X$ be non-empty and open. Then there exists an i such that $U \cap U_i \neq \varnothing$. Since U_i is irreducible, it follows that $U_i \subseteq \overline{U}$. But then it follows that in fact $U_j \cap \overline{U} \neq \varnothing$ for all j and hence since U_j is irreducible, $U_j \subseteq \overline{U}$. We conclude that $\overline{U} = X$. □

Exercise 3. *Let $X = V(xy - w^2) \subseteq \mathbb{A}^3$ be an affine surface. Prove that the blowup of X at the origin $Bl_0(X)$ is smooth.*

Answer. Notice that $\{0\} = Z(x, y, w)$ and that (x, y, w) is radical. We consider the map $\phi : X \setminus \{0\} \to X \times \mathbb{P}^2$ defined by

$$\phi(a, b, c) = ((a, b, c), (a : b : c)).$$

Then $Bl_0(X)$ is the closure of the image of ϕ. To compute this, it suffices to compute the closure in an affine cover of $X \times \mathbb{P}^2$. Let $\pi_u : X \times D_+(u) \to X \times \mathbb{A}^2$ be the canonical chart and analogously for the other coordinates v and z of $\mathbb{P}^2$. Then:

$$\pi_u \circ \phi(a, b, c) = \left(a, b, c, \frac{b}{a}, \frac{c}{a}\right)$$

where $a \neq 0$. Notice that since $X \times \mathbb{A}^2 \subseteq \mathbb{A}^5$ is closed, we may equivalently calculate the closure of $im(\pi_u \circ \phi)$ in $\mathbb{A}^5$. Writing $k[x, y, w, u, v]$ for the coordinate ring of $\mathbb{A}^5$:

$$im(\pi_u \circ \phi) \subseteq Z(xy - w^2, xu - y, xv - w).$$

In fact,

$$im(\pi_u \circ \phi) = Z(xy - w^2, xu - y, xv - w) \setminus Z(x),$$

since on this set, the map

$$(x, y, w, u, v) \mapsto (x, y, w)$$

is an inverse. Notice that by applying the last two equations to the first,

$$Z(xy - w^2, xu - y, xv - w) = Z(x^2u - x^2v^2, xu - y, xv - w).$$

We claim that the closure of $im(\pi_u \circ \phi)$ is given by

$$Z(u - v^2, xu - y, xv - w).$$

Indeed, this is a closed set, containing $Z(x^2u - x^2v^2, xu - y, xv - w) \setminus Z(x)$ and the latter set is open, dense in $Z(u - v^2, xu - y, xv - w)$.

We compute the Jacobian

$$\begin{pmatrix} 0 & 0 & 0 & 1 & -2v \\ u & -1 & 0 & x & 0 \\ v & 0 & -1 & 0 & x \end{pmatrix}.$$

Obviously, this matrix always has full rank. Hence the blowup is regular at all points contained in $D_+(u)$.

For $D_+(v)$ we observe that $xy - w^2$ is symmetric in x and y and so is the ideal (x, y, w) defining $\{0\}$. Consequently the same argument applies.

Finally, for $D_+(z)$, we proceed analogously. As above,

$$im(\pi_z \circ \phi) = Z(xy - w^2, wu - x, wv - y) \setminus Z(w).$$

Rewriting, we obtain:

$$Z(xy - w^2, wu - x, wv - y) = Z(w^2uv - w^2, wu - x, wu - y)$$

and the closure of $Z(w^2(uv - 1), wu - x, wv - y) \setminus Z(w)$ is given by

$$Z(uv - 1, wu - x, wv - y).$$

Calculating the Jacobian yields

$$J = \begin{pmatrix} 0 & 0 & 0 & v & u \\ -1 & 0 & u & w & 0 \\ 0 & -1 & v & 0 & w \end{pmatrix}.$$

This has full rank everywhere in $Z(uv - 1, wu - x, wv - y)$ since $uv - 1 = 0$ implies $u, v \neq 0$ and we conclude. $\square$