Exercise Sheet 10

Algebraic Geometry

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Exercise 1. *Lüroth's Theorem in algebra says that if L is a subfield of $k(t)$ such that $L|k$ has transcendence degree 1, then there exists an element $u \in L$ such that $L = k(u)$. Reformulate this theorem into a non-trivial statement about projective curves.*

Answer. First observe that since the transcendence degree of $k(u)$ is 1, we must have that u is transcendental over k. Consequently, the map $k[t] \to k(u)$ mapping t to u is injective. That is to say, we obtain a map of fields $k(t) \to k(u)$ sending t to u. It is obviously surjective and hence we may reformulate Lüroth's Theorem as:

Every L as in the exercise is isomorphic to $k(t)$.

We apply the equivalence of categories between regular projective curves over k and finitely generated field extensions of k of transcendence degree 1. Under this equivalence, L is mapped to some regular projective curve X and $k(t)$ is mapped to $\mathbb{P}^1$. The fact that L is a subfield of $k(t)$ means that we have a dominant rational map $\phi : X \to \mathbb{P}^1$. Now, Lüroth's theorem says that in fact L is isomorphic to $k(t)$ meaning that X is birational to $\mathbb{P}^1$. But regular projective curves are birational if and only if they are isomorphic showing that $X \cong \mathbb{P}^1$. Summarizing, we have:

Let X be a regular projective curve such that there exists a dominant rational map $\phi : X \to \mathbb{P}^1$. Then $X \cong \mathbb{P}^1$.

Exercise 2. *Show that the rational map*

$$(x, y) : Z(y^2 - x^3 - x^2) \to \mathbb{P}^1$$

does not extend to a morphism (has no regular extension to 0).

Answer. Assume such an extension ϕ would exist. Then define $a := \phi(0,0)$. We first assume that $a \in D_+(x)$. Hence composing with the map $\psi_x : D_+(x) \to \mathbb{A}^1$ given by $\psi_x(x:y) = \frac{y}{x}$ we obtain a map $f := \psi_x \circ \phi : Z(y^2 - x^3 - x^2) \to \mathbb{A}^1$. This map must be a polynomial morphism by the characterization of regular functions on affine varieties. Moreover, it is given on $Z(y^2 - x^3 - x^2) \cap D(x)$ by $f = \frac{y}{x}$. That is to say, on this open subset of $Z(y^2 - x^3 - x^2)$:

$$x \cdot f = y.$$

Since $Z(y^2 - x^3 - x^2)$ is separated (by the exercise below), the equality must hold everywhere and in $k[x,y]$ we find a $g \in k[x,y]$ such that

$$x \cdot f = y + g \cdot (y^2 - x^3 - x^2)$$

where we identify f with a representative in $k[x,y]$. But now, evaluating at $x = 0$ we obtain the equality in $k[y]$:

$$0 = y + g(0,y) \cdot y^2.$$

This is impossible (for example the derivative of the right hand side evaluated at $y = 0$ is 1 which is non-zero) and consequently, no such f can exist.

The case $a \in D_+(y)$ is analogous. □

Exercise 3. *Show that affine, projective and quasi-projective varieties over k are separated.*

Answer. We split the proof into multiple steps:

1. Every subvariety of a separated variety is separated.
2. $\mathbb{A}^n$ is separated.
3. $\mathbb{P}^n$ is separated.

Notice that these three steps show the claim, since affine, projective and quasi-projective varieties are all subvarieties of either $\mathbb{A}^n$ or $\mathbb{P}^n$.

Step 1: Let X be separated and $Y \subseteq X$. The inclusion induces a (continuous) morphism $\phi : Y \times Y \to X \times X$ given by $\phi(y,y) = (y,y)$ (that is to say, the inclusion of the products). Let Δ_X and Δ_Y denote the diagonals in $X \times X$ and $Y \times Y$. Then, since X is separated, Δ_X is closed. Moreover, $\Delta_Y = \phi^{-1}(\Delta_X)$ and by continuity of ϕ, Δ_Y is closed showing that Y is separated.

Step 2: Notice that $\mathbb{A}^n \times \mathbb{A}^n \cong \mathbb{A}^{2n}$ where $\mathbb{A}^{2n}$ has coordinate ring $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Then the diagonal is given by $Z(x_1 - y_1, \ldots, x_n - y_n)$ and is hence closed.

Step 3: It suffices to show that the intersection of Δ with every affine open subset of $\mathbb{P}^n \times \mathbb{P}^n$ is closed. Indeed, if this is the case, then its complement is open in every affine open and hence open in $\mathbb{P}^n \times \mathbb{P}^n$ showing that Δ is closed.

Hence let $D_+(x_k)$ and $D_+(y_j)$ be the corresponding affine opens in the factors $\mathbb{P}^n$. Applying the isomorphism ϕ between $D_+(x_k) \times D_+(y_j)$ and $\mathbb{A}^n \times \mathbb{A}^n$ shows that

$$\phi(D_+(x_k) \times D_+(y_j) \cap \Delta) = \Delta \subseteq \mathbb{A}^n \times \mathbb{A}^n$$

which is closed by step 2. Consequently $\mathbb{P}^n$ is separated. $\square$