# Exercise Sheet 11

Algebraic Geometry

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**Exercise 1.** *Consider the symmetric power map*

$$\phi : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \ (d \ times) \to \mathbb{P}^d.$$

*Prove that $\phi$ is a finite morphism.*

**Answer.** The symmetric power map is given as follows: Consider the polynomial

$$P(s) := \prod_{i=1}^{d} (x_i - y_i \cdot s)$$

and let $a_k$ be the coefficient to $s^k$ of $P(s)$. Then

$$\phi([x_1 : y_1], \ldots, [x_d : y_d]) = [a_0 : \cdots : a_d].$$

We aim to show that $\phi$ is projective and quasi-finite (which shows that $\phi$ is finite as shown in the lecture). To this end, we first show that $\phi$ is quasi-finite: Indeed, let $\phi([x_1 : y_1], \ldots, [x_d : y_d]) = [a_0 : \cdots : a_d]$. If $a_d = 0$, then there exists $i$ such that $y_i = 0$. But then, omitting this coordinate and replacing $d$ with $d-1$, $\phi$ restricts to the analogously defined map into $\mathbb{P}^{d-1}$ and hence we may assume that all $y_i$ are non-zero. Thus $\phi$ is in fact a map into $D_+(a_d)$. The preimage of this set under $\phi$ is precisely $D_+(y_1) \times \cdots \times D_+(y_2)$ and composing with the canonical charts of $D_+(a_d)$ and $D_+(y_i)$, we have that we can replace $\phi$ with $\psi : \mathbb{A}^1 \times \cdots \times \mathbb{A}^1 \to \mathbb{A}^d$ sending $(x_1, \ldots, x_d)$ to $(s_0, \ldots, s_{d-1})$ where $s_k$ is the $k$-th symmetric polynomial in the $x_i$. We claim that this map is in fact finite. Indeed, writing $k[x_1, \ldots, x_d]$ for the coordinate ring of $\mathbb{A}^d$, each $x_i$ satisfies the polyomial equation in $k[s_0, \ldots, s_{d-1}]$:

$$0 = \prod_{k=1}^{d} (X - x_k).$$

Consequently the $x_i$ are integral and it follows that $k[x_1, \dots, x_d]$ is a finitely generated $k[s_0, \dots, s_{d-1}]$-module showing the claim. Now, since $\psi$ is finite, so are its fibres and by the above, so are the fibres of $\phi$.

It remains to show that $\phi$ is projective. This can be seen as follows: Notice that $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is a projective variety. Indeed, considering any family of closed embeddings $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^k$ (for example the Segre embedding) and using induction, we obtain a closed embedding $i : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \to \mathbb{P}^n$ for some $n$. In particular, the map $\phi \times i : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \to \mathbb{P}^d \times \mathbb{P}^n$ is injective. Moreover, we know from the lecture that images of projective varieties under morphisms are closed and hence $\phi \times i$ is closed. It is obviously an embedding because $i$ is an embedding (the inverse "forgets" the first component and applies the inverse of $i$ ) and we conclude that $\phi \times i$ is a closed embedding. But now, $\phi = \pi \circ (\phi \times i)$ where $\pi : \mathbb{P}^d \times \mathbb{P}^n \to \mathbb{P}^d$ is the projection onto the first component and by definition $\phi$ is projective. From the lecture, we know that projective and quasi-finite morphisms are finite and we conclude. □

**Exercise 2.** *Consider the map*

$$\phi : \mathbb{P}^1 \to \mathbb{P}^1, z \mapsto [f(z) : g(z)]$$

*for some $f, g \in k[z]$ with no common zeros and not both constant. Compute the degree of $\phi$.*

**Answer.** Without loss of generality, we assume that $deg(f) \geq deg(g)$ for else, we may swap $f$ and $g$. For

Notice that $\mathbb{P}^1$ and $\mathbb{A}^1$ are birational and hence the function field of $\mathbb{P}^1$ is $k(t)$. To determine a generator, consider the birational equivalence $\phi : \mathbb{P}^1 \to \mathbb{A}^1$ defined by

$$\phi(x : y) = \frac{y}{x}$$

on $D_+(x)$. Now, the function field of $\mathbb{A}^1$ is generated by $t$ and hence we obtain a geneator of the function field of $\mathbb{P}^1$ $f := \phi^*(t)$. This is the regular function $\frac{y}{x}$ defined on $D_+(x)$. It follows that $\phi$ induces a field morphism $\phi^* : k(t) \to k(t)$ defined by

$$\phi^*(t) = \frac{g(t)}{f(t)}.$$

Consequently we must determine $[k(t) : k(g/f)]$. Equivalently, we may determine the degree of the minimal polynomial of $t$ over $k(g/f)$. First, observe that we may assume that $g$ and $f$ are monic after dividing by the lead coefficient. We claim then that the minimal polynomial is given by

$$P := f(X) - g(X) \cdot \frac{f(t)}{g(t)}.$$

This is indeed a non-constant polynomial in $X$ since $f$ is non-constant and thus the image of $P$ under the evaluation morphism sending t to some zero of $f$ is non-constant. Moreover, $t$ is a zero of $P$. It remains to show that $P$ is irreducible. We wish to apply Gauss's lemma and thus first show that $P \in k[g/f]$ is primitive: To this end, first notice that this is obvious if $deg(f) > deg(g)$. Otherwise, the leading coefficient of $P$ is $1 - f/g$. Since $f \neq g$, there exists at least one coefficient which $f$ and $g$ do not have in common. Consequently, one of the coefficients of $P$ is of the form $\alpha - \beta \cdot f/g$ where $\alpha, \beta \in k$ are not equal. Observe that:

$$\beta \cdot (1 - f/g) - (\alpha - \beta \cdot f/g) = \beta - \alpha \neq 0$$

which is a unit in $k[g/f]$ showing that these two coefficients are coprime and hence $P$ is primitive. By Gauss's lemma, it suffices to show that $P$ is irreducible in $k[g/f]$. We claim that the canonical map $\psi : k[t] \to k[g/f]$ is an isomorphism. This is equivalent to claiming that $g/f$ is not algebraic over $k$. To see that this is the case, note that $k[t]$ is not algebraic over $k$ and $k[t]$ is a finite extension of $k[g/f]$. By the multiplicativity of degrees, $k[g/f]|k$ is an infinite extension showing that $g/f$ is transcendental and the claim follows. Therefore, we may equivalently show that $Q := f(X) - g(X) \cdot t$ is irreducible in $k[t, X]$. Now, since $f$ and $g$ have no roots in common, they are coprime in $k[X]$. This means that $Q$ is primitive and applying Gauss's lemma a second time, we see that we may equivalently show that $Q$ is irreducible in $k(X)[t]$. But this is trivial since $Q$ is of degree 1 in $t$ and $k(X)$ is a field. Consequently

$$[k(t) : k(g/f)] = deg(P) = deg(f) = \max\{deg(f), deg(g)\}$$

where the second equality can be seen in the case $deg(f) = deg(g)$ by observing that the lead coefficient of $P$ is $1 - f/g \neq 0$ in this case.

**Exercise 3.** *Suppose $M$ is a $B$-module that is an increasing union of submodules $M_i$ with $M_0 = 0$ and such that every $M_{i+1}/M_i$ is a free $B$-module.*

1. *Show that $M$ is a free $B$-module.*

*We say that an arbitrary $B$-algebra $A$ satisfies $(\star)$ if for each finitely generated $A$-module $M$,there exists a nonzero $f \in B$ such that $M_f$ is a free $B_f$-module.*

2. *Suppose $B$ is a noetherian integral domain. Prove that $B$ satisfies $(\star)$.*
3. *Suppose $B$ is a Noetherian integral domain and $A$ is a finitely generated $B$-algebra satisfying $(\star)$. Prove that $A[T]$ also satisfies $(\star)$.*

**Answer.** Task 1: We show inductively that $M_i$ is free and that there exists a basis of $M_i$ which contains a basis of $M_{i-1}$:

Base case: For $i = 0$, this is trivial.

Induction step ($i \to i+1$): Assume that $M_i$ is free and let $\mathcal{B}_i$ be a basis of $M_i$. Consider the short exact sequence:

$$0 \to M_i \to M_{i+1} \to M_{i+1}/M_i \to 0.$$

Since $M_{i+1}/M_i$ is free, the sequence is split and we have

$$M_i \oplus M_{i+1}/M_i \cong M_{i+1}$$

where the isomorphism is given by $i + h$ : with $i : M_i \to M_{i+1}$ the inclusion and $h : M_{i+1}/M_i \to M_{i+1}$ some homomorphism. Now, $M_i \oplus M_{i+1}/M_i$ is free as the direct sum of free modules and we may extend the basis of $M_i$ to a basis of $M_i \oplus M_{i+1}/M_i$. Applying $i + h$, we see that we obtain a basis of $M_{i+1}$ which contains $\mathcal{B}_i$ showing the claim.

Now, define:

$$\mathcal{B} := \bigcup_{i \in \mathbb{N}} \mathcal{B}_i.$$

We claim that this is a basis of $M$. Since $M$ is the union of all $M_i$, $\mathcal{B}$ obviously generates $M$. Moreover, any non-trivial linear relation between elements of $\mathcal{B}$ would involve only finitely many basis elements and hence would yield a non-trivial relation between elements of some $\mathcal{B}_i$ for $i$ large enough. Since this is impossible, we conclude that $M$ is free with basis $\mathcal{B}$.

Task 2: We first remark that if $M_f$ is a free $B_f$-module and $g \in B$, then $M_{gf}$ is a free $B_{gf}$ module. This follows from the fact that

$$\left( \bigoplus_i B_f \right)_g \cong \bigoplus_i B_{fg}.$$

Let $M = (m_1, \ldots, m_n)$. We show inductively using task 1 that for any $B$ module with at most $n$ generators such an $f$ exists:

Base case ($n = 0$): In this case $M = 0$ and the claim is trivial.

Induction step ($n \to n+1$): Let $M_1 := (m_1, \ldots, m_n) \subseteq M = (m_1, \ldots, m_{n+1})$. By induction hypothesis, there exists $f \in B$ such that $(M_1)_f$ is a free $B_f$ module. Moreover, $M/M_1 = (m_{n+1}) = Bm_{n+1}$. Consequently the map $B \to M/M_1$, $b \mapsto bm_{n+1}$ is surjective and we have $M/M_1 \cong B/Ann(M/M_1)$. If $Ann(M/M_1) \neq \{0\}$, choose $g \in Ann(M/M_1) \setminus \{0\}$. Then notice that $(M/M_1)_g = 0$ since for every $m \in M/M_1$, $m = (gm)/g = 0/g = 0$. Hence $(M/M_1)_g$ is in particular free. If $Ann(M/M_1) = 0$, then $M/M_1 \cong B$ is free and we set $g = 1$. Now, by the above remark, both $(M/M_1)_{gf}$ and $(M_1)_{gf}$ are free $B_{gf}$-modules. Since $B$ is an integral domain, we also have $gf \neq 0$. Finally, by task 1, considering the increasing sequence $M_0 := \{0\} \subseteq (M_1)_{gf} \subseteq M_{gf}$ it follows that $M_{gf}$ is a free $B_{gf}$-module and we conclude.

Task 3: Let $M$ be a finitely generated $A[T]$-module with $M = (m_1, \dots, m_n)$. Define $M' := Am_1 + \cdots + Am_n$. Then $M'$ is a finitely generated $A$-module. Define

$$M_k := M' + T \cdot M' + \cdots + T^k M'$$

and the map of $A$-modules $\phi_k : M' \to M_k/M_{k-1}$ defined by $\phi_k(m) := T^k \cdot m$. Define $N_k$ to be the kernel of $\phi_k$. Notice that if $\phi_k(m) = 0$ then $T^k \cdot m \in M_{k-1}$ and consequently $T^{k+1} \cdot m \in M_k$ showing that the sequence $N_k$ is an increasing sequence of $A$-submodules. Since $A$ is Noetherian and $M'$ is finitely generated, there must exist some $k_0$ at which the $N_k$ stabilize. That is to say, $M_k/M_{k-1}$ are isomorphic as $A$-modules for all $k \geq k_0$. Choose $f \in B$ such that $(M_{k_0-1})_f$ is a free $B_f$ module and analogously $g \in B$ such that $(M_{k_0}/M_{k_0-1})_g$ is a free $B_g$-module. By a previous remark $(M_{k_0-1})_{gf}$ and $(M_{k_0}/M_{k_0-1})_{gf}$ are free $B_{gf}$-modules. Moreover, since $B$ is integral, $gf \neq 0$. But now, by the isomorphism above, $(M_k/M_{k-1})_{gf}$ are free modules for all $k \geq k_0$. Hence setting $N_0 := \{0\}$, $N_1 := (M_{k_0})_{gf}$, $N_2 := (M_{k_0+1})_{gf}$, etc., we see that the previous task applies and we obtain that $M_{gf}$ is a free $B_{gf}$-module. Since $M$ was arbitrary, we conclude. □