Exercise Sheet 12

Algebraic Geometry

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Exercise 1. *Describe $Spec(\mathbb{Z}[x])$ by considering fibres of $\phi : Spec(\mathbb{Z}[x]) \to Spec(\mathbb{Z})$.*

Answer. Let $q \in Spec(\mathbb{Z})$ be prime. The fibre of ϕ corresponds to all prime ideals $P \subseteq \mathbb{Z}[X]$ such that $P \cap \mathbb{Z} = q$. Hence there is a bijection between primes in the fibre over q and primes primes in:

$$\mathbb{Z}[X]/(q\mathbb{Z}[X]) \cong \mathbb{Z}/q[X].$$

whose intersection with $\mathbb{Z}/q$ is precisely 0. Let $S := (\mathbb{Z}/q \setminus \{0\}) \subseteq \mathbb{Z}/q[X]$. Since $\mathbb{Z}/q$ is an integral domain, S is a multiplicative subset. Moreover, the primes in the above ring with the condition given are precisely the primes which do not intersect S which is to say they are in bijection with the primes in

$$S^{-1}(\mathbb{Z}/q[X]) \cong (S^{-1}\mathbb{Z}/q)[X] \cong (\mathbb{Z}/q)_{(0)}[X] := A_q.$$

We discern two cases:

Case 1 ($q = (p)$ for some $p \in \mathbb{Z}$ prime): In this case, $\mathbb{Z}/q$ is a field and

$$A_q \cong \mathbb{F}_p[X].$$

This is a PID and hence the non-zero prime ideals in A_q are in bijection with irreducible polynomials in $\mathbb{F}_p[X]$. Tracing the isomorphisms above, the fibre over q is given by the ideals (p, f) where $f \in \mathbb{Z}[X]$ is irreducible in $\mathbb{F}_p[X]$ along with the ideal (p) corresponding to the zero-ideal in A_q.

Case 2 ($q = (0)$): In this case, $S = \mathbb{Z} \setminus \{0\}$ and $A_q \cong \mathbb{Q}[X]$. This is again a PID and it follows that the non-zero primes in A_q correspond to the irredicible elements in $\mathbb{Q}[X]$. By Gauss's lemma, these correspond to primitive, irreducible polynomials in $\mathbb{Z}[X]$ and the fibre over q is (f) where $f \in \mathbb{Z}[X]$ is primitive and irredicible along with the ideal (0).

Now, notice that for any p prime, $f \in \mathbb{Z}[X]$ irreducible over $\mathbb{F}_p$, $\mathbb{Z}[X]/(p,f) \cong \mathbb{F}_p$ is a field and (p,f) is maximal. Thus these points are closed, $Z(p,f) = \{(p,f)\}$. The ideals (q) satisfy that $Z(q)$ is the fibre over the ideal $(q) \subseteq \mathbb{Z}$. Finally, the ideal (0) is the generic point. This completely describes the topology of $Spec(\mathbb{Z})[X]$.

Exercise 2. *Prove that $Spec(R \times S)$ is homeomorphic to the disjoint union of $Spec(R)$ and $Spec(S)$.*

Answer. Notice that any prime in $R \times S$ is of the form $p \times S$ or $R \times q$ with $p \subseteq R$ and $q \subseteq S$ prime ideals. Namely, any ideal is of the form $I = a \times b$ with $a \subseteq R$ and $b \subseteq S$ ideals and since

$$(R \times S)/I \cong R/a \times R/b,$$

I is prime if and only if one of the factors is 0 and the other one is an integral domain. Consequently, we get a bijection

$$\phi : Spec(R) \sqcup Spec(S) \to Spec(R \times S).$$

It remains to show that it is a homeomorphism: Let $P \subseteq R \times S$ be prime. Observe that since every ideal which is not $R \times S$ is contained in a prime ideal, the sets $Z(P)$ for P prime form a basis of the Zariski topology (more precisely, their complements form a basis) and it suffices to check that $\phi^{-1}(Z(P))$ is closed. By symmetry, we may assume that $P = R \times p$. But then

$$\phi^{-1}(Z(P)) = Z(p) \subseteq Spec(S)$$

and we conclude. Conversely, let $p \subseteq S$ be prime. Then

$$\phi(Z(p)) = Z(R \times p)$$

and it follows that ϕ and ϕ^{-1} are continuous. Hence we conclude. □

An alternative more abstract argument goes as follows: Notice that *Spec* is a contravariant equivalence of categories between the category of rings and the category of affine schemes. As such, it maps products to coproducts. We conclude by observing that the product in the category of rings is the product of rings and the coproduct in the category of schemes is the disjoint union.

Exercise 3. *Let X be an affine variety over an algebraically closed field k. Let $x \in X$ be a point and m_x be the maximal ideal of the local ring $\mathcal{O}_{X,x}$. A tangent* *vector at x is an element in $(m_x/m_x^2)^\vee$. Show that there is a bijection between k-algebra homomorphisms $k[X] \to k[t]/t^2$ and pairs (x, v) where x is a point in X and v is a tangent vector at x.*

Answer. Let $f : k[X] \to k[t]/t^2$ be a k-algebra homomorphism. Notice that $k[t]/t^2$ is local with unique maximal ideal (t). Since $k[X]$ and $k[t]/t^2$ are finitely-generated k-algebras, $f^{-1}((t))$ is a maximal ideal m in $k[X]$. It corresponds to some point $x \in X$. Consequently it is suffient to show that there is a bijection between k-algebra homomorphisms $k[X] \to k[t]/t^2$ with $f^{-1}((t)) = m$ and $(m_x/m_x^2)^\vee$.

Notice that $(m_x/m_x^2)^\vee$ is a k-vector space. Since there is a bijection between any two vector spaces over k of equal dimension, it is sufficient to show that the set of k-algebra morphisms $k[X] \to k[t]/t^2$ is bijective to a vector space of equal dimension. First, we determine the dimension of $(m_x/m_x^2)^\vee$. Since the dimension is finite, this is equal the dimension of m_x/m_x^2. This in turn (by Nakayama and Krull's principal ideal theorem) is equal the (Krull) dimension of m_x. By the characterization of prime ideals in the localization, this is equal the dimension of m.

We now show that the set of k-algebra morphisms $k[X] \to k[t]/t^2$ with preimage of (t) equal to m is in bijection with $(m/m^2)^\vee$ which has the same dimension and we may conclude. Indeed, let ϕ be such a map. Then, restricting to m, we obtain a k-linear map $\overline{\phi} : m \to (t)$. Moreover, notice that $\overline{\phi}(m^2) \subseteq (t^2) = 0$. Consequently, $\overline{\phi}$ factors to a k-linear map $F(\phi) : m/m^2 \to (t) \cong k$.

Conversely, notice that $k[t]/t^2$ is a graded k-algebra isomorphic to $k \oplus k$ in degrees 0 and 1. Notice that we have a map of k-algebras $k[X] \to k \oplus m/m^2$ given by the quotient map

$$k[X] \to k[X]/m^2 \cong k \oplus m/m^2.$$

Now, if $\alpha : m/m^2 \to k$ is k-linear, it induces a k-algebra morphism $G(\alpha) := k[X] \to k \oplus m/m^2 \to k \oplus k \cong k[t]$.

It remains to show that F and G are inverse to one another. The fact that $F(G(\alpha)) = \alpha$ is clear. Indeed, the $\overline{\phi}$ constructed in the above case, corresponds to the composition of α with the projection $m \to m/m^2$ and hence the claim follows.

Conversely, $G(F(\phi))$ is a k-algebra morphism and its restriction to m is equal to the restriction of ϕ to m. Since the maximal ideal generates $k[X]$ as a k-algebra, this implies that $G(F(\phi)) = \phi$. □