Exercise Sheet 13

Algebraic Geometry

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Exercise 1. *Give an example of a morphism of ringed spaces that is not a morphism of schemes.*

Answer. Let $X = Y = Spec(R) = \{\eta, p\}$ where R is a DVR and p is the closed point. Let $f : X \to Y$ be the constant function $f(x) = p$. Now, recall that $\mathcal{O}_X(X) = R$, $\mathcal{O}_X(\{\eta\}) = K = Frac(R)$ and $\mathcal{O}_X(\varnothing) = 0$. Consequently writing $\mathcal{F} := f^*\mathcal{O}_X$, we have $\mathcal{F}(X) = R$, $\mathcal{F}(\{\eta\}) = 0 = \mathcal{F}(\varnothing)$. Define $f^\#$ by setting it to be the identity on $\mathcal{O}_X(X)$ and the zero map otherwise. This is obviously a map of sheaves and hence we obtain a map of ringed spaces $(f, f^\#)$. However, notice that $f^\#$ is not a local ring map since in fact $\mathcal{F}_\eta = 0$ (notice that $\{\eta\}$ is an open neighbourhood of η on which $\mathcal{F}$ is constantly zero) whereas $\mathcal{O}_{X,f(\eta)} = \mathcal{O}_{X,p} = R_p = R \neq 0$. We conclude that $f^\# : \mathcal{O}_{X,f(\eta)} \to \mathcal{F}_\eta$ cannot be a local ring map (in particular because the zero ring is not local) and f is not a morphism of schemes.

Exercise 2. *Show that any irreducible closed non-empty subset of a scheme has a unique generic point.*

Answer. We first show this for affine schemes: Let $X = Spec(R)$ and $Y \subseteq X$ closed and irreducible. Then $Y = Z(a)$ for some radical ideal a. We claim that in order for Y to be irreducible, a must be prime. Indeed, if not then there exist $f, g \in R$ such that $f, g \notin a$ but $fg \in a$. Since a is radical, this means that $\sqrt{a} \subsetneq \sqrt{(f + a)}$ and analogously for g. As shown in the lecture, this implies that $Z(f + a) \subsetneq Z(a)$ and thus $Z(a) = Z(f + a) \cup Z(g + a)$ is a decomposition of a into closed proper subsets showing that $Z(a)$ is not irreducible. Finally $a \neq R$

since $Y \neq \emptyset$. This shows the claim.

But now, since $\overline{\{a\}} = Z(a)$ as was shown in the lecture, a is a generic point of $Z(a)$. Moreover, if b is a second generic point then $Z(a) \subseteq Z(b)$ and thus $b \subseteq a$. But since $b \in Y = Z(a)$, we also have $a \subseteq b$ and consequently $a = b$ showing uniqueness.

For general X let $Y \subseteq X$ be irreducible. Then consider $U \subseteq X$ affine open such that $U \cap Y \neq \emptyset$. Since $U \cap Y \subseteq Y$ is open an open subset of Y and a closed subset of U, by the above argument, there exists a generic point $x \in U \cap Y$. That is to say the closure of $\{x\}$ in $U \cap Y$ is $U \cap Y$. Thus the closure of x in Y must contain $U \cap Y$. Now note that that Y is irreducible and consequently the open subset $U \cap Y$ is dense showing that in fact the closure of x in Y is Y and x is a generic point of Y. For uniqueness let $y \in Y$ be a second generic point. Observe that a point is generic if and only if it is contained in every open subset. This immediately implies that $y \in U \cap Y$. By the argument for affine schemes given above, we conclude $x = y$. □

Exercise 3. *Show that a scheme X is integral (i.e. irreducible and $\mathcal{O}_{X,x}$ is reduced for all $x \in X$) if and only if $\mathcal{O}_X(U)$ is an integral domain for any affine open subset $\emptyset \neq U \subseteq X$.*

Answer. We first show that any integral scheme satisfies that $\mathcal{O}_X(U)$ is an integral domain for any affine open subset $U \subseteq X$. To this end, we assume by contraposition that there exists an affine open subset $U \subseteq X$ such that $\mathcal{O}_X(U)$ is not an integral domain. If X is not irreducible we are done. Consequently assume X is irreducible. Notice that the stalk of $\mathcal{O}_X$ at some $x \in U$ is isomorphic to the stalk of the restriction of the sheaf $\mathcal{O}_X$ to U at x. Indeed, using the definition of the stalk by the colimit, this follows from the fact that any open subset of X contains an open subset of U. Hence in order to show that $\mathcal{O}_{X,x}$ is not reduced, we may restrict to U and thus assume $X = Spec(R)$ for some R (notice that since $U \subseteq X$ is open, it is again irreducible). We have $\mathcal{O}_X(U) = \mathcal{O}_X(X) = R$. That is to say, R is not an integral domain. Let $f, g \in R \setminus \{0\}$ with $f \cdot g = 0$. Notice that this implies that every prime must contain either f or g (since the quotient by the prime must be an integral domain) and it follows that $X = Spec(R) = Z(f) \cup Z(g)$. But now, X is irreducible and consequently it must hold that $Z(f) = X$ or $Z(g) = X$. Without loss of generality, $Z(f) = X$. That is to say, every prime contains f or equivalently, f is nilpotent. We now construct a prime p such that f is non-zero in $\mathcal{O}_{X,p}$ which finishes the proof. To do so, notice that $Ann(f) := \{r \in R : rf = 0\}$ is an ideal and since $f \neq 0$, $Ann(f) \neq R$. Consequently there exists a maximal ideal p which contains $Ann(f)$. Now, if f were zero in $\mathcal{O}_{X,p} = R_p$,then by definition of the localization, there would exist some $g \in R \setminus p$ with $fg = 0$ which is impossible since p contains the annihilator of f. Thus we conclude.

For the converse, we again proceed by contraposition and assume that X is either not irreducible or there exists an $x \in X$ such that $\mathcal{O}_{X,x}$ is not reduced.

In the first case, this is equivalent to saying that there exist $V_1, V_2 \subseteq X$ open such that $V_1 \cap V_2 = \varnothing$. Let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ be non-empty, affine subsets, $U_1 = Spec(R_1)$, $U_2 = Spec(R_2)$. Since the subsets are non-empty, the R_i are not the zero ring. By the previous exercise sheet, we now have that $\mathcal{O}_X(U_1 \cup U_2) \cong Spec(R_1 \times R_2)$. This is not an integral domain and since $U := U_1 \cup U_2$ is affine, we conclude.

In the second case, let $x \in X$ such that $\mathcal{O}_{X,x}$ is not reduced. As above, we may restrict to $X = U = Spec(R)$ affine. Then letting p be the prime corresponding to x, $\mathcal{O}_{X,x} = R_p$. Let $\frac{a}{b} \in R_p$ be nilpotent and non-zero, $a, b \in R$. Then necessarily also a is nilpotent. That is to say, there exists some $k \in \mathbb{N}$ and $g \in R \setminus p$ such that $a^k \cdot g = 0$. Hence $R = \mathcal{O}_X(U)$ is not an integral domain. $\square$

Exercise 4. *Let p be a prime. For a scheme X over $\mathbb{F}_p$, define $F : X \to X$ to be the identity map on the underlying topological space and $F^\# : \mathcal{O}_X \to \mathcal{O}_X$ to be the map $F^\#(g) = g^p$. Show that this is a map of schemes.*

Answer. Obviously, F is continuous. As remarked in the lecture, the fact that X is a scheme over $\mathbb{F}_p$ means that $\mathcal{O}_X$ is a sheaf of $\mathbb{F}_p$-algebras. Let $U \subseteq X$ be open. Then $F^\#$ obviously defines a ring morphism since $p = 0$ in $\mathcal{O}_X(U)$. Moreover, $F^\#$ obviously commutes with the restriction maps (it commutes with any ring morphism). Hence it remains to show that the induced map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is a local morphism. This map is again given by $g \mapsto g^p$. But now, if g is contained in the maximal ideal, then obviously so too is g^p. That is to say, $F^\#(\mathfrak{m}) \subseteq \mathfrak{m}$ and we conclude. $\square$