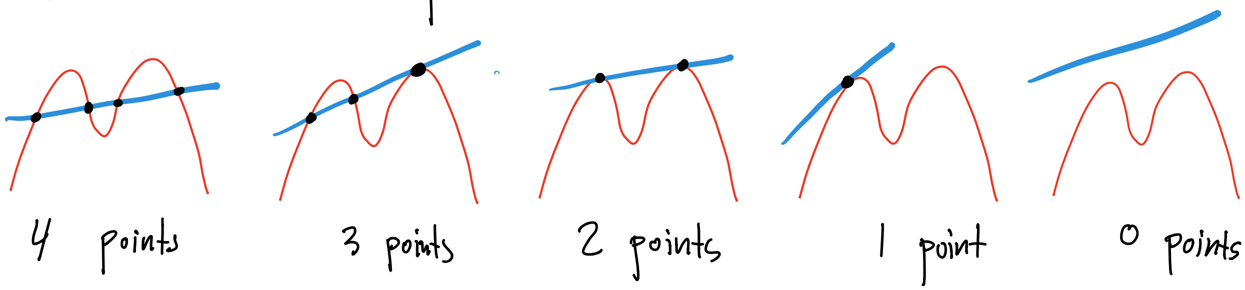
Chapter 11 - Bezout's theorem

Given two curves $C = Z(F)$ and $D = Z(G)$ of degree m and n respectively, the number of intersection points $C \cap D$ is at most mn.



However, if we

- work in $\mathbb{P}^2$
- work over an algebraically closed field
- Count intersection points with multiplicities

then we have exactly $m \cdot n$ points (Bezout's theorem)

Thm (Classical form of Bezout's theorem)

Let $Z_1, \ldots, Z_n$ be hypersurfaces in $\mathbb{P}^n$ with only finitely many points in common. Then

$$\deg Z_1 \cdots \deg Z_n = \sum_{p \in \mathbb{P}^n} \mu_p(Z_1, \ldots, Z_n)$$

multiplicity of the intersection at $p \in \mathbb{P}^n$.
(to be defined..)

Multiplicities of modules

R = noetherian graded k-algebra

M = a f.g graded R-module

Defn The multiplicity of M at $\mathfrak{p} \in \operatorname{Spec} R$ is defined by

$$\mu_{\mathfrak{p}} = \operatorname{length}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$$

← Recall: length = supremum of chains of submodules $M = M^0 \supsetneq M^1 \supsetneq \cdots \supsetneq M^d = 0$

If (R, m) is local, then

$$\mu_p(M) < \infty \iff m^d M = 0 \text{ for some } d$$

In this case,

$$\mu_p(M) = \dim_K (M \otimes_R K) \qquad K = R/m$$

Main example

X a variety

$x \in X$ a point

$I \subset \mathcal{O}_{X,x}$ an ideal with $\sqrt{I} = m \subset \mathcal{O}_{X,x}$

$I \supseteq m^r$ for some $r \gg 0$

$\rightsquigarrow \dim_k\left(\mathcal{O}_{X,x}/I\right) < \infty$ and so we define

the multiplicity of I at $x = \mu_x\left(\mathcal{O}_{X,x}/I\right) = \dim_k\left(\mathcal{O}_{X,x}/I\right)$

dimension as a k-vector sp

Affine case: $X \subseteq \mathbb{A}^n$

$$\mathcal{O}_{X,x}/I = \left(A(X)/I\right)_m$$

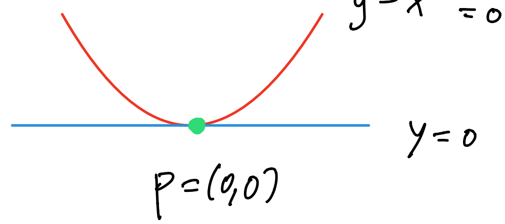
$$\rightsquigarrow \quad \mu_x\left(\mathcal{O}_{X,x}/I\right) = \mu_x\left(A(X)/I\right)$$

ex $X = \mathbb{A}^2$

$f = y - x^2$

$g = y$

$p = (0,0)$

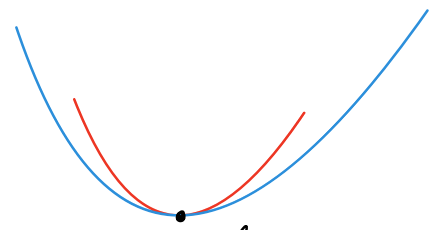


$$\rightsquigarrow \quad \mu_p\left(\mathcal{O}_{X,x} / (f,g) \right) = \dim_k \left(\left(\frac{k[x,y]}{(y, y-x^2)} \right)_{(x,y)} \right)$$

$$= \dim_k \left(\frac{k[x]}{(x^2)} \right)_{(x)}$$

$$= \dim_k \frac{k[x]}{x^2} = \underline{2.}$$

ex $X = \mathbb{A}^2$



$f = y - x^2$

$g = y - x^2 - xy$

$p = (0,0)$

$$\rightsquigarrow \quad \mathcal{O}_{X,x}/(f,g) = \left(\frac{k[x,y]}{(y-x^2,\, y-x^2-xy)}\right)_{(x,y)}$$

$$= \left(\frac{k[x,y]}{(y-x^2,\, x^3)}\right)_{(x,y)} \cong \left(\frac{k[x]}{x^3}\right)_{(x)} = \frac{k[x]}{x^3}$$

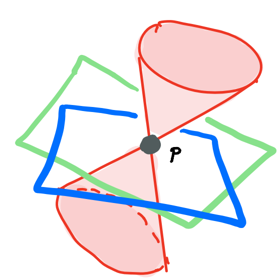
$$\therefore \quad \mu_p(\mathcal{O}/I) = 3$$

ex $X = \mathbb{A}^3$ $\quad f_1 = xy - z^2, \quad f_2 = x - z, \quad f_3 = y + x$

$p = (0,0,0)$

$$\rightsquigarrow \quad \mathcal{O}_{X,p}/(f_1,f_2,f_3) = \left(\frac{k[x,y,z]}{(xy - z^2,\, x - z,\, y + x)}\right)_{(x,y,z)}$$

$$= \left(\frac{k[x]}{x(-x) - x^2}\right)_{(x)} = \frac{k[x]}{(x^2)}$$



$\rightsquigarrow$ multiplicity $= 2$

Transverse intersections

Given $p \in \mathbb{A}^n \longleftrightarrow$ maximal ideal $m \subset k[x_1, \dots, x_n]$

For $f \in k[x_1, \dots, x_n]$ s.t. $f(p) = 0$ (so $f \in m_p$)

cotangent space

$\rightsquigarrow$ differential $df \in m/m^2$ (the class of f)

<u>Note</u>: $Z(f)$ non-singular at $p \iff df \neq 0$

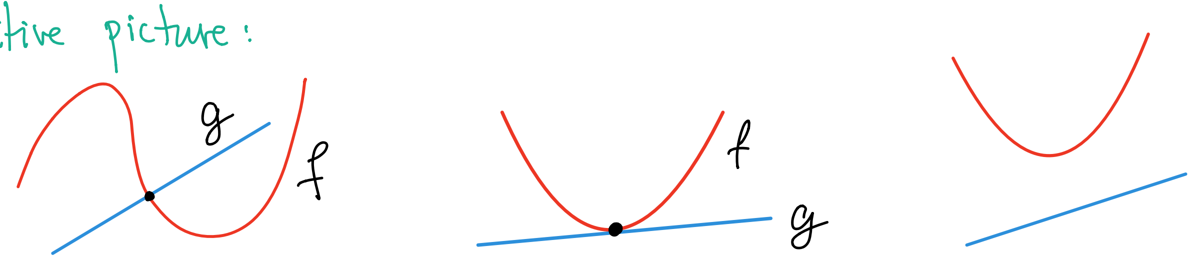
$$df = \sum \frac{\partial f}{\partial x_i}(p) \cdot x_i$$

Defn r hypersurfaces with equations $f_1, \dots, f_r$ meet *transversely* at p if

(i) they are non-singular at p; and

(ii) $df_1, \dots, df_r \in m/m^2$ are linearly independent.

Intuitive picture:



transversal — not transversal — Also transversal

$\langle df, dg \rangle = k^2$ $\qquad\qquad$ $\langle df, dg \rangle = k$

Lemma Let $A = k[t_1, \dots, t_n]_{(t_1,\dots,t_n)} = \mathcal{O}_{\mathbb{A}^n, 0}$

$f_1, \dots, f_n \in m$

TFAE:

(1) $f_1, \dots, f_n$ meet transversally at 0

(2) $(f_1, \dots, f_n) = m$

(3) $\mu_p(Z_1, \dots, Z_n) = 1$ where $Z_i = Z(f_i)$ $i = 1 \dots n$

$$df_1, \dots, df_n \text{ generate } m/m^2 \overset{\text{Nakayama}}{\Longleftrightarrow} f_1, \dots, f_n \text{ generates } m$$

$$\Updownarrow$$

$df_1, \dots, df_n$ are linearly independent

So $(1) \Leftrightarrow (2)$.

$(2) \Leftrightarrow \dim_k \left(A/(f_1, \dots, f_n)\right) = 1 \Leftrightarrow \mu_p(Z_1, \dots, Z_n) = 1$ by definition.

Rmk If, say, $f_1 \in m^2 \setminus m$, then $f_1 \dots f_n$ cannot generate the maximal ideal m

$\therefore$ If $Z(f_1)$ is singular at $p \rightarrow \mu_p \geq 2$.

Numerical polynomials

Defn A numerical polynomial is a polynomial $P \in \mathbb{Q}[z]$ s.t $P(m) \in \mathbb{Z}$ for each $m \in \mathbb{Z}$.

ex $$\binom{z}{n} := \frac{z(z-1)\cdots(z-n+1)}{n!} \in \mathbb{Q}[z]$$

is a numerical polynomial which has non-integer coefficients.

If P is a numerical polynomial, then so is $\Delta P(z)$

where $$\Delta P(z) = P(z+1) - P(z)$$

is the (forward) difference operator.

Lemma

(1) If $P \in \mathbb{Q}[z]$ and $P(m) \in \mathbb{Z}$ for all $m \gg 0$

$\rightsquigarrow$ P is a numerical polynomial

(2) $\binom{z}{n}$ form a <u>$\mathbb{Z}$-basis</u> for the group of numerial polynomials

i.e. $P(z) = c_0 \binom{z}{n} + c_1 \binom{z}{n-1} + \cdots + c_n$

for $c_0, \ldots, c_n \in \mathbb{Z}$

(3) If $f : \mathbb{Z} \to \mathbb{Z}$ is a function s.t

$\Delta f(m)$ is a polynomial for $m \gg 0$

then $\exists$ numerical polynomial $P(z) \in \mathbb{Q}[z]$ s.t

$f(m) = P(m)$ for $m \gg 0$.

Hilbert functions and Hilbert polynomials

For the rest of the section $R = k[x_0, \dots, x_n]$

All modules M are f.g. and graded.

Defn The Hilbert function of M is given by

$$h_M(i) = \dim_k M_i$$

dimension of the i-th graded part of M.

ex For $M = R$, we have

$$h_M(d) = \binom{n+d}{n}$$

ex $R = k[x_0, x_1]$ $\quad M = R/(x_0^3, x_0 x_1, x_1^5)$

$\rightsquigarrow$ h_M has values

i	0	1	2	3	4	5	6	7	8
h_M	1	2	2	1	1	0	0	0	0
		x_0, x_1	x_0^2, x_1^2	x_1^3	x_1^4				

$\rightsquigarrow$ polynomial growth for $i \geq 5$.

Theorem (Hilbert-Serre)

$I \subset R = k[x_0, \dots, x_n]$ homogeneous ideal.

$\rightsquigarrow$ unique polynomial $P_I(z) \in \mathbb{Q}[z]$ s.t

$$h_I(m) = P_I(m) \quad \text{for all } m \gg 0$$

Furthermore,

we define $\deg 0 = -1 = \dim \emptyset$

a) $\deg P_I(m) = \dim Z_+(I) \subseteq \mathbb{P}^n$

b) If $Z_+(I) \neq \emptyset$, then the leading coefficient of $P_{R/I}(z)$ is of the form $\frac{1}{(\deg P_I)!}$ (integer)

See Ellingsrud's "Lectures on Commutative Algebra".

Defn $P_I(z)$ is called the Hilbert polynomial of I.

The Degree of a variety

Defn

If $d = \dim Z_+(\operatorname{ann} M)$, we define the degree of M as

$$\deg M = d!\,(\text{leading coefficient of } P_M)$$

this is an integer.

If $X \subseteq \mathbb{P}^n$ is a closed subset, we define

$$\deg X = \deg\left(R/I(X)\right).$$

Note: If $X \neq \emptyset \Rightarrow (I(X))_d \neq R_d \quad \forall d \geq 0$

$\Rightarrow P_X \neq 0$ with positive leading coefficient

$\Rightarrow \deg X > 0.$

(If $X = \emptyset$, then $\exists\, d$ s.t. $x_i^d \in I(X)$ $\forall i$ by Nullstellensatz

$\Rightarrow \left(R/I(X)\right)_i = 0$ for $i \geq d\cdot(n+1)+1$

$\Rightarrow P_X = 0.$)

Lemma $F \in k[x_0, \dots, x_n]_d \Rightarrow X = Z_+(F) \subset \mathbb{P}^n$ has degree d.

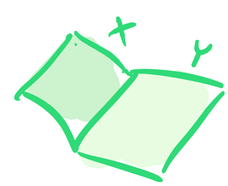
$0 \to R(-d) \xrightarrow{\cdot F} R \to R/F \to 0$ is exact

$$\rightsquigarrow h_M(z) = h_R(z) - h_R(z-d)$$

$$= \binom{z+n}{n} - \binom{z-d+n}{n} = \frac{d}{(n-1)!} z^{n-1} + \dots$$

$\Rightarrow \deg X = d$ ✓

Lemma $X, Y \subseteq \mathbb{P}^n$ of the same dimension m and with no common component.



$$\rightsquigarrow \deg(X \cup Y) = \deg X + \deg Y$$

$I_{X \cup Y} = I_X \cap I_Y$

$\rightsquigarrow$ s.e.s $0 \to R/I_{X \cup Y} \to R/I_X \oplus R/I_Y \to R/I_X + I_Y \to 0$

$$\rightsquigarrow P_{X \cup Y} = P_X + P_Y - P_{X \cap Y}$$

$X \cap Y$ has $\dim < m$ $\Rightarrow$ no contribution to leading term

$$= \frac{\deg X}{(m-1)!} z^m + \frac{\deg Y}{(m-1)!} z^m + \text{terms of lower order}$$

$\Rightarrow$ ok.

Regular sequences and Unmixedness

Defn $f_1, \dots, f_k \in R$ forms a regular sequence if f_{r+1} is not a zero-divisor modulo $(f_1 \dots f_r)$ $\forall r = 1 \dots k-1$

Note Krull's principal ideal theorem $\Rightarrow$ $\dim Z(f_1, \dots f_r) = n - r$ $\forall r = 1 \dots k-1$.

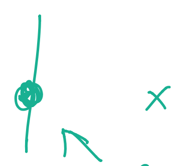
ex x^2, yz $\quad$ x^2, y^2, z^2 are regular,

x^2, xy , $\quad$ $x^2, x+y, zx + y^2$ are not.

$Z(x)$ is a component $\quad$ $Z(x,y)$ is a component $\rightsquigarrow \dim Z(f_1, f_2, f_3) = 2$

Note:

$(x^2, xy) = (x) \cap (x^2, y)$



embedded prime

$\rightarrow$ problem: there are associated prime ideals which aren't minimal.

The following result is fundamental:

The Unmixedness Theorem (Macaulay)

The polynomial ring $R = k[x_0 \dots x_n]$ is unmixed.

In other words, any ideal $I \subset R$ generated by $r = \mathrm{ht}\, I$ elements does not have any embedded primes.

$\Longleftrightarrow$ every associated prime is a minimal prime.

"I is unmixed"

Prop If $f_1, \dots, f_n \in R$ have $Z(f_1, \dots, f_n)$ finite, then $f_1, \dots, f_n$ form a regular sequence:

f_{r+1} is not a zero-divisor modulo $(f_1 \dots f_r)$ $\forall r$.

Lemma If $Z(f_1, \dots, f_n)$ is finite, then any irreducible component Z of $Z(f_1, \dots, f_r)$ has dimension $n-r$ for $r = 1, \dots, n$.

First, $\dim Z \geq n-r$ by Krull's principal ideal theorem

Similarly, $\dim(Z \cap Z(f_{r+1}, \dots, f_n)) \geq \dim Z - (n-r)$

By assumption, LHS $= 0$, so $\dim Z \leq n-r$. ✓

Cor The ideal $(f_1 \dots f_r)$ has $\operatorname{ht} = r$ and is therefore unmixed.

Now, let $M = R/(f_1, \dots, f_r)$.

All associated primes of M have ht r and $\dim \frac{M}{f_{r+1}} < \dim M$.

$\Rightarrow$ f_{r+1} is not contained in any associated prime of M

$\Rightarrow$ f_{r+1} is not a zerodivisor.

$\Rightarrow$ $f_1 \dots f_n$ is a regular sequence

Prop Let $f_1 \ldots f_n \in R$ be a regular sequence of homogeneous elements of degrees $d_1, \ldots, d_n$ respectively. Then

$$h_{R/I}(I) = d_1 \cdots d_n \qquad I = (f_1, \ldots, f_n)$$

Write $S_r = R/(f_1 \ldots f_r)$. We have sequences

$h_M(i) = \dim_k M_i$

$$0 \longrightarrow S_r(-d_{r+1}) \xrightarrow{\cdot f_{r+1}} S_r \longrightarrow S_{r+1} = S_r/f_{r+1} \longrightarrow 0$$

$f_1 \ldots f_n$ regular sequence $\Longrightarrow$ this is exact. (injective on left)

$\rightsquigarrow$ induction on r

$$h_{S_{r+1}}(z) = h_{S_r}(z) - h_{S_r}(z - d_{r+1})$$

$$= \left(\frac{d_1 \cdots d_r}{(n-r)!} z^{n-r} + a z^{n-r-1} + \cdots \right)$$

$$- \left(\frac{d_1 \cdots d_r}{(n-r)!} (z - d_{r+1})^{n-r} + a (z - d_{r+1})^{n-r-1} + \cdots \right)$$

$$= \frac{d_1 \cdots d_r}{(n-r)!} \cdot (n-r) \cdot d_{r+1} z^{n-r-1} + \cdots$$

$$= \frac{d_1 \cdots d_{r+1}}{(n-r-1)!} z^{n-r-1} + \cdots \qquad \square$$

From Hilbert polynomials to local multiplicities

Now, $I = (f_1, \ldots, f_n) \subset R$ has $\dim Z_+(I) = 0$

$\rightsquigarrow$ after a change of coordinates, we may assume
$$Z(I) \subset D_+(x_0) \subset \mathbb{P}^n$$

Lemma Let $f_1, \ldots, f_n \in k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$ denote the dehomogenizations wrt x_0, and set
$$\mathcal{O}_{Z_1 \cap \ldots \cap Z_n} = k\left[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right] \Big/ (f_1, \ldots, f_n) \overset{\text{Artinian}}{\cong} \prod_p \mathcal{O}_{Z_1 \cap \ldots \cap Z_n, p}$$

Then there is a decomposition
$$S_{x_0} = \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_{Z_1 \cap \ldots \cap Z_n} \cdot x_0^i$$

Note: $(F_1, \ldots, F_n) = (f_1, \ldots, f_n)$ in S_{x_0} $\quad (x_i^{d_i} f_i = F_i)$

The degree 0 part: Consider
$$R \to R/(F_1, \ldots, F_n) \rightsquigarrow R_{x_0} \xrightarrow{\theta} R_{x_0}/(F_1, \ldots, F_n)R_{x_0} = S_{x_0}$$
$$(\ker \theta)_0 = \left((F_1, \ldots, F_n)_{x_0}\right)_0 = \left((f_1, \ldots, f_n)_{x_0}\right)_0$$

$$\rightsquigarrow \frac{(R_{x_0})_0}{((f_1, \ldots, f_n)_{x_0})_0} \simeq (S_{x_0})_0 \Rightarrow (S_{x_0})_0 \simeq \mathcal{O}_{Z_1 \cap \ldots \cap Z_n}$$

Now, x_0 is invertible in these localizations, so
$$(S_{x_0}) \cdot x_0^i \subseteq (S_{x_0})_i$$

Conversely, any $w \in (S_{x_0})_i$ is of the form $w = a x_0^s$

where $s \in \mathbb{Z}$, $a \in (S_{x_0})_0$. Hence

$$(S_{x_0})_i = (S_{x_0})_0 \cdot x_i^i$$

$$\rightsquigarrow \quad S_{x_0} = \bigoplus_{i \in \mathbb{Z}} (S_{x_0})_0 \, x_0^i$$

$$= \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_{Z_1 \cap \dots \cap Z_n} \cdot x_0^i \qquad \square$$

Lemma For $d \gg 0$, the localization map $S \longrightarrow S_{x_0}$ induces an isomorphism

$$\rho : S_d \longrightarrow (S_{x_0})_d$$

ρ injective:

Claim $M = \ker \rho$ as an R-module has support at the origin.

$Z_+(F_1, \dots, F_n, x_0) = \emptyset \implies (F_1, \dots, F_n, x_0)$ is $\underbrace{(x_0, \dots, x_n)}_{=: m_+}$-primary

$\therefore Z(x_0)$ and $\operatorname{Supp}(S)$ have only the origin in common. in $\mathbb{A}^{n+1}$. ✓

$w \in \ker \rho \implies x_0^N w = 0$ for some $N > 0$

Also, $w \cdot F_1 = \dots = w \cdot F_n = 0$

$\implies M$ is annihilated by some power m_+^N

$\Rightarrow$ M is finite-dimensional as a k-vector space

$\Rightarrow$ $M_i = 0$ for $i \gg 0$

$\Rightarrow$ ρ is injective in large degrees.

ρ surjective

Let $\omega = a x_0^r \in S_{x_0}$ with $a \in (S_{x_0})_0$.

Modulo $(F_1, \dots, F_n)$ we may write ω as

$\omega = a x_0^r = H x_0^{r-d}$ where $H \in R_d$.

$\rightsquigarrow$ If $r \gg d$, ω lies in the image of ρ.

Now, take a basis $a_1, \dots, a_v \in (S_{x_0})_0$ (as a k-vectorspace)
and write these as $a_i = H_i x_0^{-d_j}$ where $H_i \in R_{d_i}$

$d \gg d_1 \cdots d_j \Rightarrow$ all products $a_j \cdot x_0^{d_j} \in \operatorname{im} \rho$

$$\begin{tikzcd} S_d \arrow[r, "\rho"] & (S_{x_0})_d \\ S_0 \arrow[u, "\cdot x_0^d"] \arrow[r] & (S_{x_0})_0 \arrow[u, "x_0^d", "\simeq"'] \end{tikzcd}$$

$\Rightarrow$ ρ is surjective ✓ $\square$

Prop For $d \gg 0$, the localization map $S \longrightarrow S_{x_0}$ induces an isomorphism

$$S_d \simeq \mathcal{O}_{Z_1 \cap \cdots \cap Z_n} \cdot x_0^d$$

In particular,

$$\dim_k S_d = \dim_k \mathcal{O}_{Z_1 \cap \cdots \cap Z_n}$$

Proof of Bezout's theorem

For $d \gg 0$, we have

$$
\begin{aligned}
d_1 \cdots d_n &= h_S(d) && \text{(by Hilbert polynomial computations)} \\
&= \dim S_d && \text{(def)} \\
&= \dim \mathcal{O}_{Z_1 \cap \cdots \cap Z_n} && \text{(by proposition)} \\
&= \sum_P \dim_k \mathcal{O}_{Z_1 \cap \cdots \cap Z_n, P} && \left(\mathcal{O}_{Z_1 \cap \cdots \cap Z_n} = \prod \mathcal{O}_{Z_1 \cap \cdots \cap Z_n, P} \right) \\
&= \sum_P \mu_P(Z_1, \ldots, Z_n)
\end{aligned}
$$

Basic examples

<u>Prop</u> Given $p_1, \dots, p_5 \in \mathbb{P}^5$ no 4 lying on a line

$\Rightarrow \exists$ unique conic through $p_1 \cdots p_5$.

Uniqueness: Suppose C_1, C_2 are two such conics

$\Rightarrow$ $C_1 \cap C_2 \supseteq \{p_1, \dots, p_5\}$

$\Rightarrow$ C_1 and C_2 share a component by Bezout

$\Rightarrow$ C_1 and C_2 are both reducible, say

$C_1 = Z_+(L_1 L_2)$ $\quad$ $C_2 = Z_+(L_1 L_3)$

$\Rightarrow$ $C_1 \cap C_2 = L_1 \cup (L_2 \cap L_3)$

$\Rightarrow$ $\{p_1, \dots, p_5\} \subseteq L_1 \cup \{\text{point}\}$

$\Rightarrow$ some 4 of $p_1 \cdots p_5$ lie on L_1.

Existence: Let $Q = a_{00} x_0^2 + \cdots + a_{22} x_2^2$

$Q(p_i) = 0$ $\rightsquigarrow$ linear condition on $a_{00}, a_{01}, \dots, a_{22}$

6 equations $\Rightarrow$ at least one non-zero solution $a_{00}, a_{01}, \dots, a_{22}$.

Application: Automorphisms of $\mathbb{P}^n$

Thm Any automorphism $\varphi: \mathbb{P}^n \longrightarrow \mathbb{P}^n$ is a linear transformation. That is,

$$\mathrm{Aut}\, \mathbb{P}^n = PGL_{n+1}(k) = GL_{n+1}(k) / k^*$$

We have a map

$$\rho: PGL_{n+1}(k) \longrightarrow \mathrm{Aut}\, \mathbb{P}^n$$

ρ injective:

If $M \in GL_n(k)$ induces the identity, then $M = c \cdot \mathrm{Id}_{n+1}$, $c \neq 0$. ✓

ρ surjective:

Let $\varphi: \mathbb{P}^n \longrightarrow \mathbb{P}^n$ be an automorphism.

Consider $H = Z(x_0)$.

$\rightsquigarrow$ $\varphi(H) \subset \mathbb{P}^n$ is a subvariety of codim 1

$\overset{\text{Krull}}{\rightsquigarrow}$ $\varphi(H) = Z_+(F_0)$ for some $F_0 \in k[y_0, \dots, y_n]$

Consider $\ell = Z(x_1) \cap \dots \cap Z(x_{n-1}) = Z(x_1, \dots, x_{n-1})$,

so that $H \cap \ell = (0 : \dots : 0 : 1)$

Then since φ is an automorphism,

$$\begin{aligned}\varphi((0:\cdots:0:1)) &= \varphi(H \cap \ell) \\ &= \varphi(H) \cap \varphi(\ell) \\ &= Z_+(F_0) \cap \varphi(Z(x_1)) \cap \cdots \cap \varphi(Z(x_{n-1}))\end{aligned}$$

Hence

$$1 = (\deg F_0) \cdot (\deg \varphi(Z(x_1))) \cdot \cdots \cdot (\deg \varphi(Z(x_{n-1})))$$

This means that $F_0 = a_{00}y_0 + \cdots + a_{0n}y_n$ is a linear form.

Also, we get that φ induces an isomorphism

$$\varphi|_{D_+(x_0)} : \overset{\simeq \mathbb{A}^n}{D_+(x_0)} \longrightarrow D_+(F_0) \simeq \mathbb{A}^n$$

Repeating the argument for $H = Z(x_i)$ for $i = 1 \ldots n$ shows that φ induces isomorphisms

$$\varphi|_{D_+(x_i)} : \overset{\simeq \mathbb{A}^n}{D_+(x_i)} \longrightarrow D_+(F_i) \simeq \mathbb{A}^n$$

$\varphi|_{D_+(x_0)} : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ is given by

$$\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \mapsto \left(f_1\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right), \dots, f_n\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)\right)$$

Since φ sends hyperplanes to hyperplanes, we see that the f_i must be <u>linear</u> polynomials, and thus φ must be induced by a linear map $\mathbb{P}^n \to \mathbb{P}^n$.