Chapter 12
Schemes
§ Structure sheaf

def.-Prop: The structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ is a sheaf of rings on $\operatorname{Spec} R$ s.t.:

1) $\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f \quad \forall f \in R$
2) $\mathcal{O}_{\operatorname{Spec} R, x} = R_{p_x} \quad \forall x \in \operatorname{Spec} R.$

similar to varieties

stalks

Proof sketch:

1) define $\mathcal{O}$ as a presheaf on $\{D(f)\}_{f \in R}$ given by $\mathcal{O}(D(f)) = R_f$.

Since different $f \in R$ may give the same $D(f)$, you define $\mathcal{O}(D(f)) := S_{D(f)}^{-1} R$,
where $S_{D(f)} := \{ s \in R \mid s \notin p \quad \forall p \in D(f) \}$ ← „saturation of $\{f^n\}_n$", depends only on $D(f)$, not f
and check that $R_f \to S_{D(f)}^{-1} R$ is an isom.

The restriction maps are localizations:

$D(g) \subseteq D(f) \Rightarrow S_{D(f)} \subseteq S_{D(g)} \rightsquigarrow S_{D(f)}^{-1} R \xrightarrow{\rho} S_{D(g)}^{-1} R.$

2) show that $\mathcal{O}$ is a sheaf of rings on $\{D(f)\}_{f \in R}$.

That can be formulated as follows.

Let $D(f) = \bigcup_{i \in I} D(f_i)$ be any open cover.

There are localization maps

$\rho_i : R_f \to R_{f_i}, \quad \rho_{ij} : R_{f_i} \to R_{f_i \cdot f_j}.$

Then $\mathcal{O}$ being a sheaf on $\{D(f)\}$ is equivalent to the following sequence being exact:

$$0 \to R_f \xrightarrow{\alpha} \prod_i R_{f_i} \xrightarrow{\beta} \prod_{i,j} R_{f_i \cdot f_j},$$ „sheaf exact sequence"

where $\alpha(a) = p_i(a)$ and $\beta((a_i))_{i,j} = (p_{ij}(a_i) - p_{ji}(a_j))$.

That means, by definition, that:

- α is injective (locality)

„sections agree locally $\Rightarrow$ agree globally"

- $\operatorname{Ker} \beta = \operatorname{Im} \alpha$ (gluing)

„sections agreeing on overlaps can be glued"

The proof is an algebraic exercise in localizations of rings

3) Define $\mathcal{O}_{\operatorname{Spec} R}$ to be the unique sheaf extending $\mathcal{O}$ from $\{D(f)\}_{f \in R}$ $\leftarrow$ basis of Zar. top.

Rem. 1) On other opens U, the value of $\mathcal{O}$ is less explicit, expressed via a „limit":

$$\mathcal{O}_{\operatorname{Spec} R}(U) = \lim_{D(f) \subseteq U} \mathcal{O}(D(f)) \simeq \lim_{D(f) \subseteq U} R_f$$

2) As for regular functions on a variety, the stalk at x is given by

$\bigsqcup_{U \ni x;\, f \in \mathcal{O}(U)} \{(U, f)\}/\sim$, this construction is called „colimit":

$$\mathcal{O}_{\operatorname{Spec} R, x} = \operatorname{colim}_{U \ni x} \mathcal{O}_{\operatorname{Spec} R}(U) = \operatorname{colim}_{D(f) \ni x} \mathcal{O}(D(f)) = \operatorname{colim}_{f \notin p_x} R_f = R_{p_x}.$$

"lim generalizes $\cap$, colim generalizes $\cup$"
to the situation when $R_f \to R_g$ are not injective
(lim and colim are defined via universal properties)

3) $\forall\, U \subseteq \operatorname{Spec} R$ open,

$$\mathcal{O}_{\operatorname{Spec} R}(U) \text{ is an } R\text{-algebra.}$$

Indeed, for $a \in R$ we define

$$[a] : R_f \xrightarrow{\cdot a} R_f \quad \text{on } D(f),$$

and that induces an R-module structure $\forall U$
which gives a map of sheaves

$$[a] : \mathcal{O}_{\operatorname{Spec} R} \to \mathcal{O}_{\operatorname{Spec} R}.$$

Ex. 1) $X = \operatorname{Spec} \mathbb{Z}$

$\mathcal{O}_X(D(p)) = \mathbb{Z}[\frac{1}{p}]$ $\leftarrow$ invert all $\{p^n\}_n$

$\mathcal{O}_{X,p} = \mathbb{Z}_{(p)}$ $\leftarrow$ invert all except $\{p \cdot m\}_m$

$\mathcal{O}_{X,(0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$

2) $X = \operatorname{Spec} D$, D a dvr, $m = (t)$, $K := \operatorname{Frac} D$

Denote $X = \{\underbrace{x}_{\text{closed pt}}, \underbrace{\eta}_{\text{generic pt}}\}$.

We have:

$\mathcal{O}_X(\emptyset) = 0$; $\quad \mathcal{O}_X(X) = D$; $\quad \mathcal{O}_X(\eta) = D_t = K$

$\mathcal{O}_{X,x} = D_{(t)} = D$; $\quad \mathcal{O}_{X,\eta} = D_{(0)} = K$.

§Affine schemes

We need to define ringed spaces in a more general context, when $\mathcal{O}_X$ can't be thought of as some k-valued functions for a fixed field k.

def. A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a top. space and $\mathcal{O}_X$ a sheaf of rings on X.

A morphism of ringed spaces is a pair $(f, f^\#)$ where $f: X \to Y$ is continuous and $f^\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a map of sheaves of rings on Y, where

$f_* \mathcal{O}_X(U) := \mathcal{O}_X(f^{-1}U)$ ← direct image sheaf

That means, $\forall\, U \subseteq Y$ open we have extra data of a ring hom

$$f^\#(U): \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U),$$

s.t. for $V \subseteq U$ ring homs $f^\#(-)$ are compatible with restrictions ρ_{UV}:

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r, "f^\#(U)"] \arrow[d, "\rho_{UV}"'] & \mathcal{O}_X(f^{-1}U) \arrow[d, "\rho_{f^{-1}U, f^{-1}V}"] \\ \mathcal{O}_Y(V) \arrow[r, "f^\#(V)"] & \mathcal{O}_X(f^{-1}V) \end{tikzcd}$$

is commutative.

In particular, $(f, f_\#)$ is an isom if f is a homeo and $\forall\, U \subseteq Y$ open $f^\#(U): \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$ is an isom.

Rem. 1) $f^\#$ generalizes pullback of regular functions on k-varieties:
$$h: \underset{Y}{U \subseteq} \to k \quad \rightsquigarrow \quad h \circ f : f^{-1}U \to k,$$
but we need extra data of $f^\#$ because, unlike with k-varieties, we don't have such pullback maps for free.

2) $\forall\, x \in X$, $y = f(x)$ there's an induced map
$$f^\#_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$
that sends (s, V), $y \in V \subseteq Y$ open to $(f^\# s, f^{-1}V)$, where $x \in f^{-1}V \subseteq X$ open; the map respects $\sim$ because $f^\#$ commutes with ρ_{UV}.

Since the definition of $f^\#$ is more general, we have too much freedom on the choice of $f^\#$, so we'll introduce a restriction.

def. A locally-ringed space is a ringed space $(X, \mathcal{O}_X)$ such that $\forall$ point $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring.

A morphism of locally-ringed spaces is a morphism of ringed spaces s.t. $\forall x \in X$, $y = f(x)$ the induced map

$$f_x^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

is a local hom, i.e. $f_x^\#(m_y) \subseteq m_x$, or equivalently, $(f_x^\#)^{-1}(m_x) = m_y$.

For k-varieties, this condition was automatically satisfied, because

$$m_y = \{ f \in \mathcal{O}_{Y,y} \mid f(y) = 0 \} \Rightarrow f^* m_y \subseteq m_x.$$

Main Ex. $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Prop. A ring hom $\varphi : R \to S$ induces a map of locally ringed spaces

$\operatorname{Spec} \varphi = (\varphi^*, \varphi^\#) : (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \longrightarrow (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ that satisfies:

1) on distinguished opens, $\forall f \in R$

$$R_f = \mathcal{O}_{\operatorname{Spec} R}\, D(f) \xrightarrow[\varphi^\#(D(f))]{} \mathcal{O}_{\operatorname{Spec} S}\, D(\varphi(f)) = S_{\varphi(f)}$$

is the localization of φ at f: $\frac{a}{f^n} \mapsto \frac{\varphi(a)}{\varphi(f)^n}$

2) on stalks, $\forall \mathfrak{p} \in \operatorname{Spec} S$ the induced map

$$\varphi^\#: R_{\varphi^{-1}(\mathfrak{p})} \longrightarrow S_{\mathfrak{p}}$$

is the localization of φ.

Proof sketch:

- define $h^\#$ on $D(f)$ as in 1)
- check compatibility with ρ_{uv}
- compute $h^\#$ on stalks as in 2)

def. An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$.

Affine schemes form a subcategory AffSch of locally ringed spaces.

We get a functor $\operatorname{Spec}: \mathrm{Ring}^{op} \to \mathrm{AffSch}$.

We also have a functor in the other direction, which sends $(X, \mathcal{O}_X)$ to $\mathcal{O}_X(X) =: \Gamma(X, \mathcal{O}_X)$

global sections

and $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to $f^\#(Y): \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$.

Thm. The functor

$$\boxed{\mathrm{Spec}: \ \mathrm{Ring}^{op} \xrightarrow{\sim} \mathrm{AffSch}}$$

is an equivalence, with inverse functor Γ.

In particular, $f: \mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is an isom of locally ringed spaces iff $f^{\#}: R \xrightarrow{\sim} S$ is an isom.

Proof: enough to show that for $X = \mathrm{Spec}\, S$, $Y = \mathrm{Spec}\, R$,
$f: X \to Y \in \mathrm{AffSch} \rightsquigarrow \mathrm{Spec}(\Gamma(f)) = f$.

Let $\varphi := \Gamma(f) = f^{\#}(Y): R \to S$; let $x \in X \leftrightarrow q \subset S$, $f(x) \leftrightarrow p \subset R$.

Want: $f = \mathrm{Spec}\, \varphi$ & $f^{\#} = (\mathrm{Spec}\, \varphi)^{\#}$.

We have a commutative diagram:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & S \arrow[d] \\ R_p \arrow[r, "f^{\#}_x"'] & S_q \end{tikzcd}$$

$\leftarrow$ localizations $\rightarrow$

Hence $\varphi(R - p) \subseteq S - q$, so $\varphi^{-1}(q) \subseteq p$.
However $f^{\#}_x$ is local, so $\varphi^{-1}(q) \supseteq p$.
$\Big\} \Rightarrow \varphi^{-1}(q) = p$,
so $\mathrm{Spec}\, \varphi = f$ as maps of top. spaces.

From the diagram we also get that $\forall x$ the stalk map $f^{\#}_x$ equals the localization of φ, i.e. $(\mathrm{Spec}\,\varphi)^{\#}_x$, because the map $R \xrightarrow[\varphi]{} S \to S_q$ factors uniquely through R_p.
(Similarly $f^{\#}(D(h)): R_h \to S_{\varphi(h)}$ is the localization of φ $\forall h \in R$.)
Hence maps of sheaves $\mathrm{Spec}\,\varphi^{\#}$ and $f^{\#}$ coincide.

Reminder: $X = \operatorname{Spec} R$

1) $D(f) = \{x \in X \mid f(x) \neq 0\} \rightsquigarrow$

$$\mathcal{O}_X(D(f)) = R_f$$

we allow to invert powers of f because they do not vanish on $D(f)$

2) $$\mathcal{O}_{X,x} = \left\{ (U, f) \;\middle|\; \begin{array}{l} x \in U \subseteq X \text{ open} \\ f \in \mathcal{O}_X(U) \end{array} \right\} \Big/ \sim \quad \rightsquigarrow$$

$$\mathcal{O}_{X,x} = R_{\mathfrak{p}_x}$$

germs of functions encode local behaviour around x, hence we allow to invert all functions that don't vanish at x, i.e. $R - \mathfrak{p}_x$.

Ex: $X = \operatorname{Spec} \mathbb{Z}$

$$\mathcal{O}_X(D(p)) = \mathcal{O}_X(\operatorname{Spec}\mathbb{Z} - (p)) \cong \mathbb{Z}[\tfrac{1}{p}] \cong \left\{ \frac{m}{p^n} \right\}$$

$$\mathcal{O}_{X,(p)} = \mathbb{Z}_{(p)} = \left\{ \frac{m}{l},\ p \nmid l \right\}$$

§ Schemes

def: A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme, i.e. $X = \bigcup_{i \in I} U_i$ open cover, s.t.
$\forall i \ \exists$ ring R_i: $(U_i, \mathcal{O}_X|_{U_i}) \cong (\operatorname{Spec} R_i, \mathcal{O}_{\operatorname{Spec} R_i})$.

$x \in X \rightsquigarrow$ the stalk $\mathcal{O}_{X,x}$ is the **local ring at x**.
If $x \in U = \operatorname{Spec} R \subseteq X$ open, then

$$\mathcal{O}_{X,x} = \mathcal{O}_{U,x} = R_p, \quad p = p_x.$$

Residue field at x:
$m \subset \mathcal{O}_{X,x} \rightsquigarrow \kappa(x) := \mathcal{O}_{X,x}/m \cdot \mathcal{O}_{X,x} = R_p/p \cdot R_p$.

A **morphism** or a map of schemes is a map $(f, f^\#)$ of locally ringed spaces.

Schemes form a category $\mathrm{Sch} \supset \mathrm{AffSch}$.
One can compute maps to affine schemes similar to the case of varieties.

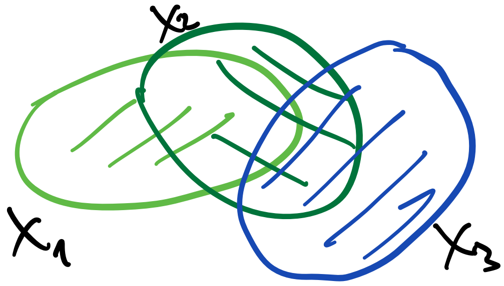
Thm. $\forall X$ scheme, R ring

$$\mathrm{Maps}_{\mathrm{Sch}}(X, \operatorname{Spec} R) = \mathrm{Maps}_{\mathrm{Ring}}(R, \mathcal{O}_X(X)).$$

(for $X = \operatorname{Spec} S$, see above; in general: similar proof)
Hence giving a map $X \to \operatorname{Spec} R$ is equivalent to giving an R-algebra structure to $\mathcal{O}_X$.

Gluing: how to get non-affine schemes

Idea:



X_i – schemes that „agree on intersections"
$\Rightarrow X = \bigcup X_i$ is a scheme

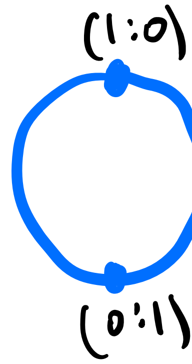
Roughly speaking, on $X_i \cap X_j$ you should specify gluing isoms

$$\mathcal{O}_{X_i}|_{X_i \cap X_j} \xrightarrow{\sim} \mathcal{O}_{X_j}|_{X_i \cap X_j}.$$

Ex.: projective line

two charts

$\rightsquigarrow$



$\mathbb{P}^1_{\mathbb{Z}}$ ← can be any ring R

variable

$U_0 = \operatorname{Spec} \mathbb{Z}[u] \simeq \mathbb{A}^1$, $\quad U_1 = \operatorname{Spec} \mathbb{Z}[u^{-1}] \simeq \mathbb{A}^1_{\mathbb{Z}}$

$U_0 \supset U_{01} := D(u) = \operatorname{Spec} \mathbb{Z}[u^{\pm 1}] \quad U_1 \supset U_{10} := D(u^{-1}) = \operatorname{Spec} \mathbb{Z}[u^{\pm 1}]$

We glue U_0 and U_1 along $U_{01} \simeq U_{10}$ and get the projective line!

Indeed, for $a \neq 0$ $\quad [1:a] = [\frac{1}{a}:1]$

coordinates in different charts

§Integral schemes

def. A scheme $(X, \mathcal{O}_X)$ is reduced if all local rings are reduced (no nilpotents).

Exercise: $\mathcal{O}_{X,x}$ reduced $\forall x \in X \iff \mathcal{O}_X(U)$ reduced $\forall$ (affine) open $U \subseteq X$.

In particular, Spec R is reduced iff R is reduced.

Associated reduced scheme:

$\operatorname{Spec} R_{red} \hookrightarrow \operatorname{Spec} R$, where $R_{red} := R/_{\mathrm{Nil} R}$

closed immersion

same top. spaces, different structure sheaves!

Ex: $R = k[t]/_{t^n} \rightsquigarrow \operatorname{Spec} R_{red} = \operatorname{Spec} k \hookrightarrow \operatorname{Spec} R$

For any scheme X, one can glue $X_{red} \longrightarrow X$, and it is universal: for a reduced scheme Y any map $Y \to X$ factors through X_{red}.

def. A scheme is integral if it is reduced and irreducible.

Exercise: $(X, \mathcal{O}_X)$ is integral iff $\mathcal{O}_X(U)$ is an int. domain $\forall$ (affine) open $U \subseteq X$.

Spec R integral $\iff$ $\mathrm{Nil} R = (0)$ & $\mathrm{Nil} R$ is prime $\iff (0)$ is prime.

Structure sheaf of an integral scheme

We can think of sections of $\mathcal{O}_X$ as of certain rational functions!

def. X integral scheme, $\eta \in X$ the generic pt.
The **function field** of X is
$$\kappa(X) := \mathcal{O}_{X,\eta}.$$

It is a field because $\forall\, \operatorname{Spec} R \subseteq X$ open
$$\mathcal{O}_{X,\eta} = \mathcal{O}_{\operatorname{Spec} R,\eta} = R_{(0)} = \operatorname{Frac} R$$
integral domain

Prop: X integral, $U \subseteq X$ open, $\eta \in X$ generic pt.

1) The canonical map
$$\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = \kappa(X) \quad \text{is injective}$$

2) $V \subseteq U$ open $\Rightarrow$ the restriction map
$$\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V) \subseteq \kappa(X) \quad \text{is injective}$$

3) $\forall\, x \in X \quad \mathcal{O}_{X,x} \subseteq \kappa(X)$, and
$$U \ni x \Rightarrow \mathcal{O}_X(U) \subseteq \mathcal{O}_{X,x}$$

4) $$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x} \subseteq \kappa(X)$$

If $X = \operatorname{Spec} R$, $\mathcal{O}_X(U) = \left\{ f \in \kappa(X) \;\middle|\; \forall\, x \in U \;\; f = \frac{g}{h},\ h(x) \neq 0,\ g,h \in R \right\}$

can pick a representative, it depends on x

Proof: 1) Let $f \in \mathcal{O}_X(U)$, assume $f(\eta) = 0$. Then $\forall$ affine open $V = \operatorname{Spec} S \subseteq U$ we have $\rho_{UV}(f) = 0$, because S is an integral domain $\Rightarrow S \hookrightarrow \operatorname{Frac} S = k(X)$.

Take $U = \bigcup_{i \in I} V_i$ affine open cover $\Rightarrow$ $\rho_{UV_i}(f) = 0 \ \forall i \Rightarrow f = 0$ because $\mathcal{O}_X$ is a sheaf.

2) The inclusions $\mathcal{O}_X(U) \hookrightarrow k(X)$ are compatible with restriction maps ρ_{UV}:

$$\begin{tikzcd} \mathcal{O}_X(U) \arrow[rr, "\rho_{UV}"] \arrow[dr, hook] & & \mathcal{O}_X(V) \arrow[dl, hook'] \\ & \mathcal{O}_{X,\eta} & \end{tikzcd}$$

commutes $\Rightarrow \rho_{UV}$ is injective.

3) we have a canonical map

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,\eta}$$
$$[U, f] \longmapsto [U, f]$$

It's injective: $\mathcal{O}_{X,x} = \mathcal{O}_{V,x}$ for $x \in \underset{\text{open nbhd}}{V} = \operatorname{Spec} A \Rightarrow$

$$\mathcal{O}_{X,x} = \mathcal{O}_{V,x} = A_p \hookrightarrow \operatorname{Frac} A = \mathcal{O}_{X,\eta}$$

for A an integral domain.

For $U \ni x$ the map $\mathcal{O}_X(U) \hookrightarrow k(X)$ factors through $\mathcal{O}_{X,x} \Rightarrow$

$$\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_{X,x}.$$

4) by 3), $\mathcal{O}_X(U) \subseteq \cap \mathcal{O}_{X,x}$

Let $f \in \cap \mathcal{O}_{X,x} \subseteq k(X)$.

Then $\forall x$ $\exists$ open nbhd $x \in V(x) \subseteq U$:
$f \in \mathcal{O}_X(V(x))$.

Since $U = \bigcup_x V(x)$, we can glue a function $f \in \mathcal{O}_X(U)$ because the values agree on all $V(x) \cap V(x')$ since they coincide inside $k(X)$.

The last equality follows because
$X = \operatorname{Spec} R \rightsquigarrow \mathcal{O}_{X,x} = R_{\mathfrak{p}_x}$.

§ Varieties vs schemes

Schemes are more useful than varieties!

Examples from our course:

① projective schemes ($\subseteq \mathbb{P}^n$) can be conveniently built from graded rings:

$\mathbb{P}^n_R = \mathrm{Proj}\, R[t_1, \dots, t_n]$

analogue of Spec for graded rings

And blow-ups are better defined this way:

$\mathrm{Bl}_{\mathrm{Spec} A/I} \mathrm{Spec} A := \mathrm{Proj} \bigoplus_{n \geq 0} I^n$; $\quad E \cong \mathrm{Proj} \bigoplus_{n \geq 0} I^n / I^{n+1}$.

② intersection theory is done using schemes: multiplicities of intersection pts can be encoded by scheme-theoretic fibers, which in turn have nilpotents.

solving classical problems in enumerative geometry!

def. $S \in \mathrm{Sch}$. An S-scheme is a scheme X with a chosen map $X \to S$, called structure morphism.

A morphism of S-schemes is a comm. diagr.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

$\rightsquigarrow$ define the category Sch_S.

Abbreviate: $\mathrm{Sch}_{\mathrm{Spec} A} =: \mathrm{Sch}_A$.

Ex. $S = \operatorname{Spec} A \rightsquigarrow X \in \mathrm{Sch}_S$ means giving an A-algebra structure to $\mathcal{O}_X(X)$.

In particular, $\mathrm{Sch} = \mathrm{Sch}_{\mathbb{Z}}$.

def. A k-scheme X is of finite type if $X = \bigcup_{j=1}^{m} \operatorname{Spec} A_j$ for some fin. gen. k-algebras A_j.

Assume $k = \bar{k}$ (till the end of the chapter).

We know:

$$X \longmapsto \operatorname{Spec} k[X]$$

affine k-varieties $\simeq$ reduced, finite type affine k-schemes

$X \mapsto k[X]$ $\qquad$ $\wr$ $\qquad$ $\wr$ $\qquad$ $\operatorname{Spec} R \mapsto R$

$\left(\begin{array}{c}\text{reduced fin. gen.}\\ k\text{-algebras}\end{array}\right)^{op}$

Goal: $k = \bar{k} \Rightarrow$

$$\boxed{k\text{-varieties} \simeq \text{reduced, finite type } k\text{-schemes}}$$

Q.: How to get a scheme from a (non-affine) variety?

A.: Make it sober! :)

def. A top. space is sober if every irreducible closed subset has a unique generic point ($\forall Z \ \exists! \, \eta_Z : \overline{\eta_Z} = Z$).

Ex: 1) Underlying top. space of any scheme (for affines - proved before, in general - exercise)

2) Underlying top. space of a k-variety is not sober: all points are closed (as maximal ideals).

def. X top. space. Its soberification is a sober top. space $S(X)$ with a map $X \to S(X)$ that is universal: $\forall \ X \xrightarrow{f} Y$ with Y sober factors as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ S(X) \arrow[ur, dashed, "\exists! S(f)"'] & \end{tikzcd}$$

I.e. $\exists$ functor $S: \mathrm{Top} \to \mathrm{Top}^{sob}$, left adjoint to the inclusion $\mathrm{Top}^{sob} \subseteq \mathrm{Top}$.

Explicitly, $S(X)$ is constructed as follows:

- points of $S(X)$ are the closed irred subsets of X
- open sets in $S(X)$ correspond to open sets in X:

$U \subseteq X$ open $\rightsquigarrow$ $U' := \{\text{irr closed subsets of } X \text{ that meet } U\} \subseteq S(X)$

declare to be open in $S(X)$

Lemma.

1) there's a bijection between $\mathrm{Open}(X)$ and $\mathrm{Open}(S(X))$

$U \subseteq X$ open $\leftrightarrow U' \subseteq S(X)$ open

Moreover, $U' \simeq S(U)$.

2) this bijection preserves $\bigcap_{i=1}^{m}$ and $\bigcup_{i \in I}$, hence $\mathrm{Open}(S(X))$ form a topology and for sheaves we get $\mathrm{Shv}(X) \underset{\text{equiv. of cats.}}{\simeq} \mathrm{Shv}(S(X))$

3) The canonical map:

$$X \longrightarrow S(X) \qquad x \longmapsto \{\overline{x}\}$$

is continuous

4) $S(X)$ is sober and satisfies the universal property

Proof: 1) $U \not\subseteq V \Rightarrow \exists\, u \in U,\ u \in V^c \Rightarrow \overline{u} \subseteq V^c \Rightarrow U' \not\subseteq V'$

Consider $S(U) \longrightarrow S(X)$ $\quad \{Z\} \longmapsto \{\overline{Z}\}$

$$\begin{array}{ccc} & \cup\!\!\mid & \\ \dashrightarrow & U' & \end{array} \qquad \begin{array}{c} \cap \\ S(U) \end{array} \qquad \begin{array}{c} \cap \\ U' \subseteq S(X) \end{array}$$

The other map is:

$$S(U) \longleftarrow U' \qquad \{W \cap U\} \longleftarrow\!\shortmid \{W\}$$

and these are inverse continuous maps.

2) ok

3) under $X \to S(X)$ the preimage of U' is U

($x \notin U \Rightarrow \overline{x} \cap U = \emptyset$ because $U \subseteq X$ open)

4) exercise

Ex. X affine k-variety $\Rightarrow$

$$\operatorname{Spec} k[X] = S(X) \text{ as a top. space.}$$

Indeed, points of $\operatorname{Spec} k[X]$ are $\mathfrak{p} \subset k[X]$, and they correspond to irreducible closed subsets of X.

Topologies are compatible:

closed subsets of $S(X)$ are given by

$$Z \subseteq X \text{ closed} \leftrightsquigarrow \left\{ \begin{matrix} \{T\} \in S(X) \\ T \subseteq Z \end{matrix} \right\}$$

When $X \subseteq \mathbb{A}^n$ k-variety, $Z \subseteq X$ closed

$$\{ T \subseteq Z \mid T \subseteq X \text{ closed irred} \} \leftrightsquigarrow \{ \mathfrak{p} \supseteq I(Z) \mid \mathfrak{p} \subset k[X] \text{ prime} \}$$

the corresponding closed subset of $\operatorname{Spec} k[X]$

Prop. 1) $(X, \mathcal{O})$ k-variety $\Rightarrow$ $(S(X), \mathcal{O})$ is a k-scheme

2) this gives a functor

$$S : \mathrm{Var}_k \longrightarrow \mathrm{Sch}_k$$

$$f : X \to Y \longmapsto S(f) : \begin{array}{ccc} S(X) & \longrightarrow & S(Y) \\ \cup\!\shortmid & & \cup\!\shortmid \\ \{Z\} & \longmapsto & \{\overline{f(Z)}\} \end{array}$$

Proof: 1) X affine $\Rightarrow S(X) = \operatorname{Spec} k[X]$, and $\mathcal{O}_X$ gives $\mathcal{O}_{S(X)}$ because $\operatorname{Shv}(X) = \operatorname{Shv}(S(X))$.

$X = \bigcup U_i$ affine cover $\Rightarrow$ $S(X)$ is some ringed space, which has a cover $S(X) = \bigcup \operatorname{Spec} k[U_i]$ $\Rightarrow$

$S(X)$ is a scheme with $\mathcal{O}_{S(X)} \leftrightsquigarrow \mathcal{O}_X$,

and $\mathcal{O}_{S(X)} \leftrightsquigarrow \mathcal{O}_X$ is a sheaf of k-algebras.

2) $S(f): S(X) \to S(Y)$ is a map of top. spaces.
The map of sheaves $S(f)^{\#}: \mathcal{O}_{S(Y)} \to S(f)_* \mathcal{O}_{S(X)}$ is obtained by replacing $\mathcal{O}_{S(Y)} \rightsquigarrow \mathcal{O}_Y$, $\mathcal{O}_{S(X)} \rightsquigarrow \mathcal{O}_X$ and $f^{\#}: \mathcal{O}_Y \to f_* \mathcal{O}_X$ is defined via precomposition with f.
$S(f)$ is a k-scheme map because $f^{\#}$ preserves k-algebra structure.

Rem. X a k-variety $\Rightarrow$
$X \to S(X)$ is a subspace embedding, because all points of X are closed.

Prop. X k-variety, $p \in S(X) \rightsquigarrow \underset{\substack{\text{closed}\\ \text{irreducible}}}{Z \subseteq X}$.
Then $k(Z) = \kappa(p)$.
In particular, X irreducible $\Rightarrow$
$$k(X) = \kappa(\eta) = \kappa(S(X)).$$
By definiton of the function field of an integral scheme

Proof for X irred, $Z = X$:
$$k(X) = \bigcup_{\substack{U \subseteq X \text{ open} \\ f \in \mathcal{O}_X(U)}} (U, f) /\sim \;= \bigcup_{\substack{U' \subseteq S(X) \\ f \in \mathcal{O}_{S(X)}(U')}} (U', f) /\sim \;= \kappa(\eta)$$
η is the generic pt of $S(X)$

$Z \subseteq X$ irreducible: exercise (good to do!)

Claim: $X \subset S(X)$ equals $\{x \in S(X) \mid \kappa(x) = k\} \subset S(X)$.
called k-points

Proof: $0 = \operatorname{tr} \deg_k \kappa(x) \overset{\text{Prop.}}{=} \operatorname{tr} \deg_k k(x) \iff x$ is a 0-dim. closed irred subvar. of X, i.e. a point.
Points of X have $\kappa(x) = k$ because $\kappa(x)/k = \bar{k}$ is finite.

Thm. The functor
$$S : Var_k \to Sch_k$$
is fully faithful, and its essential image is given by reduced finite type k-schemes.

Proof (fully faithfulness):

$f: X \to Y \in Var_k \rightsquigarrow S(f): S(X) \to S(Y)$.

Let $h: S(X) \to S(Y)$, want: $h = S(f)$ for $f: X \to Y$.

To show: $h(X) \subseteq Y$, so $h|_X : X \to Y$.

Can be checked locally $\Rightarrow$ assume $S(X), S(Y)$ affine $\Rightarrow$ h is induced by a hom of finite type k-algebras $\Rightarrow$ by Nullstellensatz the preimage of a max ideal is a max ideal
$\Rightarrow$ by Claim, h sends points of X to points of Y.

Moreover, $h|_X : X \to Y$ is a map of k-varieties and if $h = S(f)$ then $h|_X = f$ by construction.

Ess. image: X k-variety $\Rightarrow S(X)$ reduced finite type$/k$.
Let W be a reduced fin. type k-scheme, define $X := \left\{ \begin{matrix} \text{closed pts} \\ \text{of } W \end{matrix} \right\} \subset W$.

residue field of all pts in U

To get a variety: we have $\mathcal{O}_{S(X)}(U') \xrightarrow{\varphi_U} C(U, k)$
it commutes with restrictions $\rightsquigarrow \mathcal{O}_X(U) := \varphi_U(\mathcal{O}_{S(X)}(U'))$
X affine $\rightsquigarrow$ this is the structure sheaf (check on $D(f)$)
$\Rightarrow (X, \mathcal{O}_X)$ is a variety.

X is a subspace of $\mathcal{W}$ and a k-variety.

$\mathcal{W}$ sober $\Rightarrow$ get a map $S(X) \to \mathcal{W}$.

Claim (skip proof): $S(X) \to \mathcal{W}$ is an isom of schemes.