# Chapter 13
# Weil conjectures

**This chapter is NOT part of the exam!**

Enjoy :)

(following Hartshorne, Appendix C + Milne „Lectures on étale cohomology", chapter I)

**Main idea:** intrinsic connections

arithmetic of alg. varieties over finite fields $\longleftarrow\text{-----}\longrightarrow$ topology of alg. varieties over complex numbers

Explaining these connections uses many new words, but I will give some idea.

**Scalar extension**

rings $\rightsquigarrow$ affine schemes $\rightsquigarrow$ schemes

$X \in Sch_R$, $S$ an $R$-algebra $\rightsquigarrow$ $X \times_R S = \operatorname{Spec} k[x] \otimes_R S$ (for $X$ affine)

**Ex:** 1) $\mathbb{A}^n_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} = \operatorname{Spec} \mathbb{R}[t_1, .., t_n] \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{A}^n_{\mathbb{C}}$

2) $X \in Sch_{\mathbb{Z}}$ (similarly for $X \in Sch_{\mathbb{Z}_{(q)}}$)

$\nearrow X \times_{\mathbb{Z}} \mathbb{F}_q \in Sch_{\mathbb{F}_q}$, $q = p^m$

$\searrow X \times_{\mathbb{Z}} \mathbb{C} \in Sch_{\mathbb{C}}$

„look at solutions of equations over different fields"

**def.** A $k$-scheme $X$ is **smooth** if $\overline{X} := X \times \overline{k}$ is a regular variety.
(using $Var^k_{\overline{k}} \hookrightarrow Sch_{\overline{k}}$)

## § Topological invariants

Betti realization: $Sch_{\mathbb{C}} \longrightarrow Top$

$X \longmapsto X^{an} := X(\mathbb{C})$ — „analytification" (an); $X(\mathbb{C})$: points of $X$ with residue field $\mathbb{C}$

$\mathbb{A}^n_{\mathbb{C}} \longmapsto \mathbb{C}^n$ with complex topology

restricts to smooth schemes: $Sm_{\mathbb{C}} \longrightarrow Mfd_{\mathbb{C}}$

$X^{an} \in Top \rightsquigarrow H^*(X^{an}, \mathbb{Z})$ - singular cohomology

**def.** The **Betti numbers** of $X$ are

$$B_i(X) := rk\ H^i(X^{an}, \mathbb{Z}) \qquad i = 0 \dots 2n \quad (n = \dim X^{an})$$

The **Euler characteristic** of $X$ is

$$E(X) := \sum_i (-1)^i B_i$$

**Ex:** 1) $X = \mathbb{P}^1_{\mathbb{C}} \rightsquigarrow X^{an}$ is the Riemann sphere

$B_0 = B_2 = 1,\ B_1 = 0 \Rightarrow E = 2$

2) $P \in Top$ polyhedron $\Rightarrow$

$$V - E + F \overset{(1)}{=} E(P) \overset{(2)}{=} 2$$

(1) cellular cohom = singular cohom
(2) $P$ is homeo to $S^2$

**Rem.** $X \in Sch_k$ ⇝ still can define $E(X)$, less explicitly than for $k = \mathbb{C}$.

## § Arithmetic invariants

$X$ finite type $\mathbb{F}_q$-scheme.

$N_r(X) := |X(\mathbb{F}_{q^r})|$ – number of pts of $X$ with residue field $\mathbb{F}_{q^r}$

If $X \subseteq \mathbb{A}^n_{\mathbb{F}_q}$ is cut out by some equations with $\mathbb{F}_q$-coefficients, then $N_r(X)$ counts solutions whose coords are in $\mathbb{F}_{q^r}$.

**Ex:** $X = \mathbb{P}^1_{\mathbb{F}_q}$

$N_r(X) = q^r + 1 \quad \forall r$, because

$$\mathbb{P}^1(\mathbb{F}_{q^r}) = \bigcup_{a \in \mathbb{F}_q^r} \{[1:a]\} \cup \{[0:1]\}$$

**Q:** How to pack $\{N_r(X)\}_r$ nicely together?
**A:** Get inspiration from the $\zeta$-function!

**def.** The zeta-function of $X$ is

$$Z(X;t) := \exp\left(\sum_{r\geq 1}^{\infty} N_r \frac{t^r}{r}\right) \in \mathbb{Q}[[t]]$$

$\exp(f) = 1 + f + \frac{f^2}{2!} + \dots$

**Ex:** $X = \mathbb{P}^1_{\mathbb{F}_q} \Rightarrow Z(\mathbb{P}^1;t) = \exp\left(\sum_{r=1}^{\infty}(q^r+1)\frac{t^r}{r}\right) = \frac{1}{(1-t)(1-qt)}$

## § Weil conjectures

**Thm.** $X$ smooth projective $\mathbb{F}_q$-scheme of dimension $n$.

1) $Z(t) := Z(X;t)$ is a rational function:
$Z(t) \in \mathbb{Q}(t) \subset \mathbb{Q}[[t]]$

2) $Z(t)$ satisfies a functional equation:
$$Z\left(\frac{1}{q^n \cdot t}\right) = \pm q^{n \cdot E/2} \cdot t^E \cdot Z(t),$$ where $E = E(X)$

3) analogue of the Riemann hyp.:
$$Z(t) = \frac{P_1(t) \cdot P_3(t) \cdot \ldots \cdot P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdot \ldots \cdot P_{2n}(t)}$$

where $P_i(t) \in \mathbb{Z}[t]$ $\forall i$:
$P_0(t) = 1-t$; $P_{2n}(t) = 1-q^n t$ and
$\forall\, 1 \leq i \leq 2n-1$ all roots of $P_i(t)$
are algebraic integers with
absolute value $q^{-\frac{i}{2}}$.

4) We have: $E(X) = \sum (-1)^i \deg P_i$.

Moreover, if $X = Y \times_{\mathbb{Z}} \mathbb{F}_q$ for $Y$ a smooth $\mathbb{Z}$-scheme,
(or $\mathbb{Z}_{(q)}$-scheme)
then
$$\deg P_i = B_i(Y \times_{\mathbb{Z}} \mathbb{C})$$

Arithmetic data over $\mathbb{F}_{q^r}$ ($N_r$) is related to topological data over $\mathbb{C}$ ($E$ and $B_i$)!

Ex: 1) $X = \mathbb{P}^1_{\mathbb{Z}} \rightsquigarrow P_1(t) = 1 \rightsquigarrow \deg P_1(t) = 0 = B_1$.

2) $X = \mathbb{P}^n_{\mathbb{Z}} \rightsquigarrow Z(X,t) = \dfrac{1}{(1-t)(1-qt)\cdot\ldots\cdot(1-q^n t)}$,

and all the properties follow (exercise).

## Relation to Riemann hypothesis

$\zeta$-function: $\zeta(s) = \prod_p \dfrac{1}{1-p^{-s}}$

We have: $p = |\mathbb{F}_p| = |k(x)|$ for $x = (p) \in \mathbb{Z}$.

$X$ finite type $\mathbb{Z}$-scheme $\rightsquigarrow$

$$\zeta_X(s) := \prod_{\substack{x \in X \\ \text{closed pts}}} \frac{1}{1-|k(x)|^{-s}}$$

↖ finite fields for closed pts $x$

Ex. $\zeta_{\operatorname{Spec}\mathbb{Z}}(s) = \zeta(s)$

Claim: If $X$ is a finite type $\mathbb{F}_q$-scheme, then it's a f.t. $\mathbb{Z}$-scheme (via $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{F}_q$) and

$$\zeta_X(s) = Z(X; q^{-s}).$$

(to check, take log of both sides and use basic Galois theory: $L/k$ separable ext. $\Rightarrow \#(L \hookrightarrow \bar{k})_{/k} = [L:k]$ )

**Speculation:** If Spec $\mathbb{Z}$ was an $\mathbb{F}_q$-curve ("$\mathbb{F}_1$")

3) $\Rightarrow$ roots of $Z(t)$ = roots of $P_1$ = alg integers with absolute value $q^{-\frac{1}{2}}$.

$\Rightarrow$ by Claim, roots of $\zeta(s)$ have \$ $\mathrm{Re} = \frac{1}{2}$ \$ \$ \$ \$ \$ \$ \$ \$

(trivial zeroes aren't visible here because Spec $\mathbb{Z}$ is not projective, one would need to compactify...)

However, Spec $\mathbb{Z}$ is not an $\mathbb{F}_q$-scheme, in particular $\zeta(s)$ is not rational and its roots are not algebraic integers.

But one can dream to use alg-geom methods to prove the Riemann hypothesis... :)

## § History

- Weil proved the conjectures for $\mathbb{F}_q$-curves
- Weil: conjectures would follow formally if there was a cohomology theory for $\mathbb{F}_q$-schemes with nice properties similar to singular cohomology of $\mathbb{C}$-schemes, and comparable with $H^*_{sing}(X^{an}, \mathbb{Z})$ when $X$ is a $\mathbb{Z}$-scheme.

- Alg Geom Seminar (SGA) at IHES (Grothendieck, Serre, Artin, Deligne, ...) constructed $\ell$-adic cohomology with expected properties ⇝ proved (1), (2), (4)

Methods turned out to be more important than the statement, as it often happens!

- Deligne: proved (3) – much harder
  „Riemann hyp."

(for the proof he wrote up SGA 4.5 and disappointed Grothendieck who wanted to deduce ③ from his Standard Conjectures — still wide open...)

# § ℓ-adic cohomology

$\ell \neq p$ prime

$X$ finite type $k$-scheme, $k$ alg closed field of char $p$

$\mathbb{Z}_\ell := \lim\limits_{r} \mathbb{Z}/\ell^r$ – ring of ℓ-adic integers (dvr)

$\mathbb{Q}_\ell := \operatorname{Frac} \mathbb{Z}_\ell$ – not alg closed; $\exists\ \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$

$H^*(X;\mathbb{Q}_\ell)$ – ℓ-adic cohomology, built out of étale cohomology of $X$ (in fact defined for all schemes)

## Properties

① $H^*(X;\mathbb{Q}_\ell)$ are fin. dim. $\mathbb{Q}_\ell$-vector spaces, $H^i(X;\mathbb{Q}_\ell)=0$ unless $0\leq i\leq 2\cdot\dim X$.

② $f: X\to Y \rightsquigarrow f^*: H^*(Y;\mathbb{Q}_\ell)\to H^*(X;\mathbb{Q}_\ell)$

③ there's a cup-product and Poincaré duality

④ $\dim H^i(X;\mathbb{Q}_\ell)$ is constant under scalar extension to $K \supseteq k$, and for $X$ a smooth projective $\mathbb{C}$-scheme

$$H^i(X;\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}\mathbb{C} \simeq H^i(X^{an};\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}$$ „comparison theorem"

⑤ „Lefschetz fixed point formula"

$X$ smooth projective $k$-scheme, $f: X\to X$ morphism with isolated fixed pts „of multiplicity 1" $\Rightarrow$

$\#$ fixed pts of $f = \sum_i (-1)^i \mathrm{Tr}(f^*; H^i(X;\mathbb{Q}_\ell))$

trace of a linear map

Ex: $X$ = discrete finite set

$H^* = H^0 = \bigoplus_{x\in X} \mathbb{Q}_\ell \cdot \{x\}$ => the matrix of $f^*: H^0 \to H^0$ on the diagonal has 1 for each fixed pt and 0 otherwise => $\#$ fixed pts = trace

Main idea: $X$ smooth projective $\mathbb{F}_q$-scheme, $\dim X = n$.

$\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q} \rightsquigarrow F: \overline{X} \to \overline{X}$ — Frobenius map

fixed pts of $F$ = pts of $X$ with coords in $\mathbb{F}_q$ i.e. $\kappa(x) = \mathbb{F}_q$

fixed pts of $F^r$ = pts with residue field $\mathbb{F}_{q^r}$.

$$N_r(X) = \#\text{ fixed pts of } F^r = \sum_{i=0}^{2n} (-1)^i \cdot \mathrm{Tr}(F^{r*}, H^i(\overline{X}, \mathbb{Q}_\ell)).$$

Hence: $$Z(X;t) = \prod_{i=0}^{2n} \left[\exp\left(\sum_{r=1}^{\infty} \mathrm{Tr}(F^{r*}, H^i(\overline{X};\mathbb{Q}_\ell)) \cdot \frac{t^r}{r}\right)\right]^{(-1)^i}.$$

$\parallel$ $\det(1 - F^* \cdot t; H^i)^{-1}$ ← like in characteristic poly

example: $\exp\left(\sum_{r=1}^{\infty} \lambda^r \frac{t^r}{r}\right) = \frac{1}{1-\lambda t}$

$$\Rightarrow \quad Z(X,t) = \frac{P_1(t) \cdot \ldots \cdot P_{2n-1}(t)}{P_0(t) \cdot \ldots \cdot P_{2n}(t)}$$

where $P_i(t) = \det(1 - F^* \cdot t; H^i(\overline{X}; \mathbb{Q}_\ell))$.

We get Weil conjectures except „Riemann hyp.":

(1) – ok, because $\mathbb{Q}[[t]] \cap \mathbb{Q}_\ell(t) = \mathbb{Q}(t)$

(2) – follows formally from Poincaré duality

(4) – follows from comparison theorem

(3) does not follow ⇝ Deligne's Fields medal!