# Chapter 2
# The Zariski topology

## § Zariski topology

Last time:

- affine space $\mathbb{A}^n$: $k^n$ as a set, top. space - today!
- (closed) algebraic set: zero locus of an ideal

$\mathfrak{a} \subseteq k[x_1, \dots, x_n] \rightsquigarrow Z(\mathfrak{a}) := \{x \in \mathbb{A}^n \mid f(x) = 0 \ \forall f \in \mathfrak{a}\}$

- coordinate ring: $X \subseteq \mathbb{A}^n$ alg set $\rightsquigarrow k[X] = k[x_1, \dots, x_n]/I(X)$

Main idea: algebraic sets are building blocks for algebraic varieties same way as open balls are building blocks for manifolds.

In Russian, French, Italian, Japanese, ...
„variety" = „manifold"
fr. variété      ger. Mannigfaltigkeit (Riemann)

An alg variety is locally an alg set.

To make sense of „locally" we introduce Zariski topology, which is well designed to work with algebraic sets, with the cost of being rather counterintuitive.

**def.-Prop.** The Zariski topology on $\mathbb{A}^n$:
closed subsets are algebraic sets
(open sets are their complements).

**Proof:** check axioms of top. space

1) $\emptyset = Z(1)$ and $\mathbb{A}^n = Z(0)$ are closed sets
2) $Z(\mathfrak{a}) \cup Z(\mathfrak{b}) = Z(\mathfrak{a} \cap \mathfrak{b})$ is a closed set
3) $\bigcap_{i \in I} Z(\mathfrak{a}_i) = Z(\sum_{i \in I} \mathfrak{a}_i)$ is a closed set

**def.** The Zariski top. on $X$, for $X \subseteq \mathbb{A}^n$ an algebraic set, is the topology induced from Zariski top. on $\mathbb{A}^n$.

$$\text{Closed sets in } X = Z(\mathfrak{a}) \cap X$$
$$\parallel$$
$$Z(\mathfrak{a}) \text{ for } \mathfrak{a} \supseteq I(X) = Z(\mathfrak{a}) \text{ for } \mathfrak{a} \subseteq k[X]$$

From Hilbert's Nullstellensatz we get:

**Prop.** There is an inclusion-reversing bijection

$$\text{closed subsets in } X \longleftrightarrow \text{radical ideals in } k[X]$$

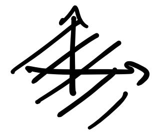
Ex. ① $X = \mathbb{A}^1$

Closed sets = finite sets (and $\mathbb{A}^1$) $\Rightarrow$ non-empty open sets are of the form
$\mathbb{A}^1 \setminus \{p_1, \dots, p_r\}$. (it's the cofinite topology).

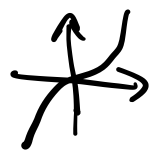
② $X = \mathbb{A}^2$

Building blocks of closed sets:

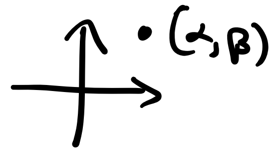
- $Z(0) = \mathbb{A}^2$



- $Z((f(x,y)))$



- $Z((x-\alpha, y-\beta))$

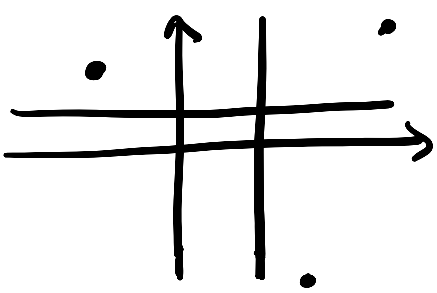


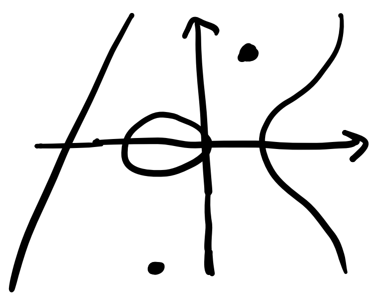

③. $X = \mathbb{A}^1 \times \mathbb{A}^1$ (product topology)

Isomorphic to $\mathbb{A}^2$ as a set, but not as a top. space!

closed sets in $\mathbb{A}^1 \times \mathbb{A}^1$ $\subset$ closed sets in $\mathbb{A}^2$

closed sets in $\mathbb{A}^1 \times \mathbb{A}^1$ $\not\supset$ closed sets in $\mathbb{A}^2$





NB: 1) say, $k = \mathbb{C}$

Then $k^n = \mathbb{C}^n$ also has Euclidean topology $\Rightarrow$ every alg. set $X$ has an induced topology (called complex topology).

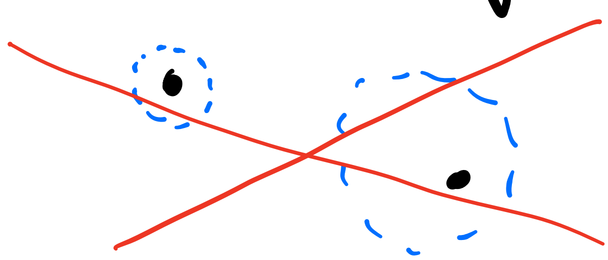
Complex top. is very different from Zariski top.!

$$\underset{\text{open in complex top.}}{U \subseteq k^n} \quad \underset{\not\Rightarrow}{\Leftarrow} \quad \underset{\text{open in Zariski top.}}{U \subseteq k^n}$$

Open balls $B_r$ are NOT open in $\mathbb{A}^n$!

2) Even the example $X = \mathbb{A}^1$ shows that Zariski topology is far from being Hausdorff: $U_1, U_2$ non-empty open subsets in $\mathbb{A}^1 \Rightarrow U_1 \cap U_2 \neq \emptyset$.

So, this topology is „counterintuitive":



does not look this way!

Nonetheless, we use it because it's defined over any field (e.g., when char $k > 0$) and it's good for encoding information about zeroes of systems of polynomial equations.

**def.** A distinduished open set in $X$ is a subset of the form
$$D(f) := \{x \in X \mid f(x) \neq 0\} \text{ for some } f \in k[X].$$

Indeed, $D(f) = X - Z(f)$ is open in $X$.

**Prop.** $X$ closed alg set. Then $\{D(f)\}$ form a basis for the Zariski topology on $X$.

**Proof:** Let $U \subseteq X$ be open.

Want: $U = \bigcup_i D(f_i)$ for some $f_i \in k[X]$.

Since $U^c$ is closed,

$U^c = Z(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq k[X]$.

Let $\mathfrak{a} = ((f_i)_{i \in I})$, then

$Z(\mathfrak{a}) = \bigcap_i Z(f_i) \Rightarrow U = Z(\mathfrak{a})^c = \bigcup_i D(f_i)$.

# § Irreducible components

**def.** A top. space is irreducible if it is not a union of two proper closed subsets.

Equivalently, a space is irreducible if the intersection of any two non-empty open subsets is non-empty (otherwise $X = U_1^c \cup U_2^c$).

**Ex.** • irred $\qquad \mathbb{R}^n = \{|x| \leq 1\} \cup \{|x| \geq 1\}$ not irred

**Lemma.** $X$ irreducible. Then:

1) $\underset{\neq \emptyset}{U} \subset X$ open $\Rightarrow U$ irreducible and dense ($\overline{U} = X$)

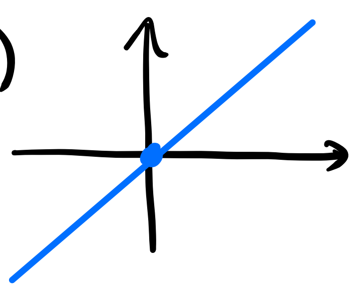
2) $X \subset X' \Rightarrow \overline{X}$ is irreducible

3) $f: X \to Y$ continuous $\Rightarrow f(X)$ is irreducible

**Proof:** follows from the equivalent description, e.g.

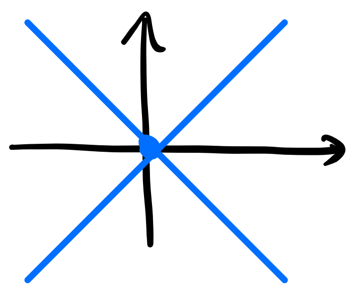
3) $U_1, U_2 \subset f(X)$ non-empty open $\Rightarrow$
$f^{-1}(U_1), f^{-1}(U_2) \subset X$ non-empty open $\Rightarrow$
$f^{-1}(U_1 \cap U_2) = f^{-1}(U_1) \cap f^{-1}(U_2) \neq \emptyset \Rightarrow U_1 \cap U_2 \neq \emptyset$.

**Ex.** 1)



$Z(x-y)$
irreducible

2)



$Z((x+y)\cdot(x-y)) = Z(x-y) \cup Z(x+y)$
not irreducible

**Prop.** An algebraic set $X \subseteq \mathbb{A}^n$ is irreducible iff $\mathfrak{a} := I(X)$ is **prime**.
In particular, $\mathbb{A}^n = Z(0)$ is irreducible.

**Proof.** ($\Rightarrow$) if $f \cdot g \in \mathfrak{a}$, then
$X \subseteq Z(f) \cup Z(g) \Rightarrow Z(f) \cap X = X$ or $Z(g) \cap X = X$
$\Rightarrow X \subseteq Z(f)$ or $X \subseteq Z(g) \Rightarrow f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.

($\Leftarrow$) if $X = Z(\mathfrak{a}') \cup Z(\mathfrak{a}'') = Z(\mathfrak{a}' \cap \mathfrak{a}'')$
with $\mathfrak{a}', \mathfrak{a}''$ radical, then $\mathfrak{a} = \mathfrak{a}' \cap \mathfrak{a}''$
$\Rightarrow \mathfrak{a} = \mathfrak{a}'$ or $\mathfrak{a} = \mathfrak{a}''$ because $\mathfrak{a}$ is prime
$\Rightarrow X = Z(\mathfrak{a}')$ or $X = Z(\mathfrak{a}'')$.

## Algebro-geometric dictionary

$X$ alg closed set, $k$ alg closed. We get a bijection:

| algebra of $k[X]$ | geometry of $X$ |
|---|---|
| maximal ideals | points |
| prime ideals | irreducible closed subsets |
| radical ideals | closed subsets |

Algebra and geometry shed light on each other's problems!

**def.** An algebraic set $X = Z(f)$, for $f \in k[x_1, \dots, x_n]$ is called a **hypersurface**.
If $f$ is linear, it's a **hyperplane**.

**Cor.** Irreducible hypersurfaces correspond to (powers of) irreducible polynomials.

**Ex.** $GL_n \subset M_{n\times n} \simeq \mathbb{A}^{n^2}$ is the complement of a hypersurface $\{\det = 0\}$.
$GL_n$ is open subset of $\mathbb{A}^{n^2}$ $\Rightarrow$ irreducible.
In fact, $\{\det = 0\}$ is irreducible too.

~~Recall~~ Noether-Lasker thm (comm alg):
every ideal in a Noetherian ring (e.g. $k[x_1,\dots,x_n]$)
is $$\mathfrak{a} = q_1 \cap \dots \cap q_r, \quad q_i \not\supset \bigcap_{j\neq i} q_j,$$
where $q_i$ are primary ideals, i.e. $\sqrt{q_i}$ are prime.
The decomposition is unique modulo embedded components, in particular, the associated primes $\sqrt{q_i}$ are unique.

This translates to the geometric language:

**Thm.** Any closed algebraic set has a decomposition
$$X = X_1 \cup \dots \cup X_r, \quad X_i \not\subset \bigcup_{j\neq i} X_j \ \forall i,$$
where $X_i$ are irreducible closed subsets.
The subsets $X_i$ are maximal irreducible subsets, they are called irreducible components of $X$.
The decomposition is unique up to reordering.
This is an example of a more general phenomenon.

def. A topological space $Y$ is Noetherian if every chain of closed subsets

$$\dots Y_{i+1} \subseteq Y_i \subseteq \dots \subseteq Y_0 = Y$$

stabilizes eventually ($Y_j = Y_N$, $j \geq N$).

Hilbert's basis thm

Ex. $\forall$ alg. set $X$ is noetherian: $k[X] = k[x_1, \dots, x_n]/I(X)$ is a Noetherian ring $\Rightarrow$ ascending chains of ideals in $k[X]$ stabilize, and they correspond to descending chains of closed subsets of $X$.

Thm $X$ Noetherian top. space, $Y \subseteq X$ closed subset $\Rightarrow$

$$Y = Y_1 \cup \dots \cup Y_r, \quad Y_i \not\subseteq \bigcup_{j \neq i} Y_j$$

s.t. $Y_i$ are irreducible. The decomposition is unique. They are irreducible components of $Y$.

Proof ($\exists$).

- $X$ Noetherian $\Rightarrow$ $\forall$ family $\Sigma \neq \emptyset$ of closed subsets has a minimal member (none of its subsets are in $\Sigma$), otherwise you can extend any descending chain
- Let $\Sigma$ be the family of closed subsets of $X$ that don't have such a decomposition, it has a minimal element $M$.

$M$ is not irreducible $\Rightarrow M = M_1 \cup M_2$, and one of them doesn't have a decomposition (otherwise $M$ would) $\Rightarrow$ $M$ was not minimal $\Rightarrow \Sigma = \emptyset$. ↯ $\square$

This argument is called <u>Noetherian induction</u>.

<u>Ex.</u> 1) hypersurfaces are the simplest:
$X = Z(f)$, $f = f_1^{a_1} \cdot \ldots \cdot f_r^{a_r}$, $f_i$ irreducible $\Rightarrow$
$X = Z(f_1) \cup \ldots \cup Z(f_r)$ is the decomposition.

2) $X = Z(\underbrace{x^2 - yz}_{f}, \underbrace{xz - x}_{g}) \subset \mathbb{A}^3$

$$\begin{tikzcd}
 & & y=0 \\
g=0 \arrow[r, squiggly] \arrow[rd, squiggly] & x=0 \arrow[ru, squiggly, "f=0"'] \arrow[r, squiggly] & z=0 \\
 & z=1 \arrow[r, squiggly, "f=0"'] & y=x^2
\end{tikzcd}$$

We get: $X = Z(x,y) \cup Z(x,z) \cup Z(y-x^2, z-1)$.

# § Dimension

**def.** $Y$ topological space. The dimension of $Y$ is
$$\dim Y := \sup\{Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_r\}$$
where $Y_i \neq \emptyset$ are closed irreducible subsets.
Dim $= \infty$ is possible (but not for algebraic sets).

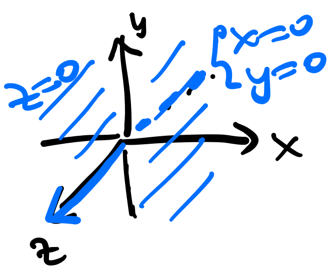
It is defined in such a way that it's compatible with Krull dimension.

**Prop.** $X$ closed alg set $\Rightarrow$ $\dim X = \dim k[X]$.
In particular, $\dim \mathbb{A}^n = n$ (comm alg fact).

**Proof** By Nullstellensatz,
$$\{X_0 \subset X_1 \subset \dots \subset X_r\} \leftrightsquigarrow \{I(X_r) \subset I(X_{r-1}) \subset \dots \subset I(X_0)\},$$
closed irred subsets of $X$ — prime ideals in $k[X]$

so suprema of lengths are the same.

**Ex.** $X = Z(z \cdot x, z \cdot y) \subseteq \mathbb{A}^3$




$$\left.\begin{array}{l} \dim Z(z) = 2 \\ \dim Z(x) \cap Z(y) = 1 \end{array}\right\} \Rightarrow \dim X = 2$$

$k[x,y,z]/_{z} \simeq k[x,y] \supset (x,y) \supset (x) \supset (0)$ 2

$k[x,y,z]/_{(x,y)} \simeq k[z] \supset (z) \supset (0)$ 1

We used that $\dim X = \max_{\text{irred comps}} \dim X_i$ (exercise).

**Prop.** $Y$ top space, $Y' \subseteq Y$ closed subspace.
Then $\dim Y' \leq \dim Y$.
If $Y, Y'$ irreducible, then $\dim Y' = \dim Y \Rightarrow Y' = Y$.

**Cor.** $X \subseteq \mathbb{A}^n$ algebraic set $\Rightarrow \dim X$ is finite ($\leq n$).

**Prop.** $Y$ top. space, $U \subset Y$ open subset.
Then $\dim U \leq \dim Y$.

**Proof:** assume $\dim U = r$, consider
$U_0 \subset U_1 \subset \dots \subset U_r$ a chain in $U$.
Then $\overline{U}_i$ are also irreducible, and $\overline{U}_i \cap U = U_i$,
so they form a strictly ascending chain in $Y$.

**Ex.** $Y = \{y, \eta\}$ where $y$ is closed and $\eta$ open (Sierpinski space)
Then $\{\eta\}$ is an open subset of dim $0$, but
$Y$ has dim $1$ thanks to the chain $\{y\} \subset Y$.
When we get to discuss schemes,
we shall see that this space corresponds
to certain rings like p-adic numbers $\mathbb{Z}_p$,
so it's important in alg geom!

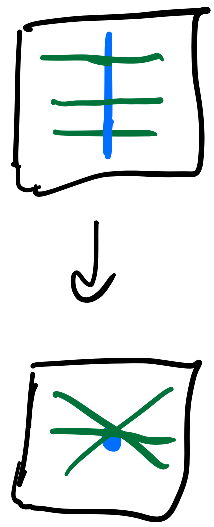
# § Polynomial maps

**def.** Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be algebraic sets.
A map $\varphi = (q_1, \dots, q_m) : X \to Y \subseteq \mathbb{A}^m$ is a *polynomial map* if all components $q_i$ are polynomial functions on $X$,
i.e. $\varphi$ is induced by a map $\mathbb{A}^n \to \mathbb{A}^m$ given by polynomials.
We denote their set $\mathrm{Hom}_{\mathrm{Aff}}(X, Y)$ ← affine

**Ex.**

$$\varphi : \mathbb{A}^2 \to \mathbb{A}^2$$
$$(x, y) \mapsto (x, xy)$$

$\mathbb{A}^2$



$$\varphi^{-1}(u,v) = \begin{cases} (u, u^{-1}v) & u \neq 0 \\ y\text{-axis} & u = 0 \ v = 0 \\ \emptyset & u = 0 \ v \neq 0 \end{cases}$$

$$\mathrm{Im}\, \varphi = \underbrace{\left(\mathbb{A}^2 - Z(x)\right)}_{\text{open}} \cup \underbrace{\{(0,0)\}}_{\text{closed}}$$ neither open nor closed!

Preimage of the lacking $x$-axis should be the "line at $\infty$" (later: projective varieties)...

**Construction.** A polynomial map $\varphi : X \to Y$ induces a $k$-algebra homomorphism

$$\varphi^* : k[Y] \to k[X].$$
$$f \mapsto f \circ \varphi$$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\varphi^* f"'] & Y \arrow[d, "f"] \\ & \mathbb{A}^1 \end{tikzcd}$$

polynomial function on $X$

**Thm** There is a contravariant equivalence of cats:

no nilpotents

algebraic sets and polynomial maps $\to$ reduced, fin. generated k-algebras and k-algebra homs

$$X \mapsto k[X]$$
$$\varphi: X \to Y \mapsto \varphi^*: k[Y] \to k[X]$$

(That is, $X \mapsto k[X]$ and $\varphi \mapsto \varphi^*$ are functorial bijections.)

**Proof.** We will show the bijections, functoriality is an exercise.

- bijection on objects:

- $\forall X$ alg. set $k[X]$ is a reduced, fin. gen. k-algebra
- any fin. gen. algebra is some $A = k[x_1, \dots, x_n]/I$.

$A$ reduced $\Rightarrow$ $I$ radical

$\Rightarrow A = k[X]$ for $X = Z(I) \subseteq \mathbb{A}^n$.

• bijection on morphisms:

– we have

$$\begin{array}{ccc} k[y_1,\dots,y_m] & \xrightarrow[\Phi^*]{} & k[x_1,\dots,x_n] \\ \downarrow & & \downarrow \\ k[Y] & \xrightarrow[\varphi^*]{} & k[X] \end{array} \qquad y_i \mapsto q_i(x_1,\dots,x_n)$$

hence we have: $\varphi^* = \psi^* \Rightarrow \varphi$ and $\psi$ have the same component functions $\Rightarrow \varphi = \psi$.

– given $\alpha : k[Y] \to k[X]$, define $\varphi : X \to Y$ by the components $q_i := \alpha(\overline{y_i})$.

$\mathbb{A}^1 \xrightarrow{\varphi} \mathbb{A}^1$
$t \mapsto t^2$

$k[t] \xleftarrow{\varphi^*} k[t]$
$f(t^2) \leftarrow f(t)$

$\alpha = 0$ on $I(Y) \Leftrightarrow \varphi(X) \subseteq Y$.

By construction, $\varphi^* = \alpha$ since they agree on generators.

**Prop.** Polynomial maps are continuous in Zariski top.

**Proof:** $\varphi : X \to Y$ polynomial

want: $\varphi^{-1}(\text{closed})$ is closed in $X$.

$\alpha \in k[Y] \rightsquigarrow \varphi^{-1}(Z(\alpha)) = Z(\varphi^* \alpha)$ is closed.

$\alpha_1,\dots,\alpha_m \in k[Y] \rightsquigarrow \varphi^{-1}(Z(\alpha_1,\dots,\alpha_m)) = \bigcap_i \varphi^{-1} Z(\alpha_i)$ is closed too.

**Rem.** 1) Zariski top. is a smallest such T1-topology.

2) not all continuous maps are polynomial, e.g. $\mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{C}}$ $z \mapsto \overline{z}$ is continuous.