Chapter 3
Sheaves and varieties

Main idea: a variety is a space that locally looks like an irreducible algebraic set.

E.g., projective plane $\mathbb{P}^2$ will be glued out of 3 copies of $\mathbb{A}^2$.

An algebraic set comes together with its Zariski topology and polynomial maps, and we want to keep track of this information $\rightsquigarrow$ category of ringed spaces.

§ Sheaves

invented by Jean Leray in prison during WWII!

def. A presheaf of sets (groups, rings, spaces, ...) on a category $\mathcal{C}$ is a functor

$$F: \mathcal{C}^{op} \to \mathrm{Set} / \mathrm{Grp} / \mathrm{Ring} / \mathrm{Top} / \dots$$

A presheaf on a top. space X is a presheaf on the category $\mathrm{Open}(X)$:

Obj = open subsets U of X

Mor = inclusions $V \subseteq U$.

This means that a presheaf R on X consists of the following data:

$\forall\, U \subseteq X$ open $\rightsquigarrow R(U)$ (set / group / ring / ...)

$\forall\, V \subseteq U$ opens $\rightsquigarrow R(U) \xrightarrow{\rho_{UV}} R(V)$ (map / hom / ...)

restriction map

s.t. $\rho_{UU} = id_{R(U)}$ and $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for $U \supseteq V \supseteq W$.

Elements of $R(U)$ are called sections, elements of $R(X)$ are global sections and $\rho_{UV}(f)$ is denoted $f|_V$.

Rem. A notion of a presheaf is very general, but we will mostly work with presheaves of k-algebras on X, so ρ_{UV} will be k-algebra homs.

Ex. 1) representable presheaf on $\mathcal{C}$

By Yoneda lemma, $\forall A \in \mathcal{C}$ gives a presheaf

$$h_A := \mathrm{Hom}_{\mathcal{C}}(-, A)$$

2) constant presheaf A_X (on X)

again, pick $A \in$ Set / Ring / Top / ...

$$A_X(U) = A \quad \forall U \text{ open in } X$$

$$\rho_{UV} = id_A \quad \forall V \subseteq U$$

3) presheaf of C^∞-functions

X smooth manifold,
$R(U) := C^\infty(U, \mathbb{R})$,
ρ_{UV} = usual restriction of functions.

Similarly, for X a complex manifold we can consider a presheaf of holom. functions.

Now, we want to be able to glue values on X from local data.

def. A sheaf R on X is a presheaf on X that satisfies:

1) $\forall\ U = \bigcup_i U_i \subseteq X$ open cover; $s, t \in R(U)$
$s|_{U_i} = t|_{U_i}\ \forall i \Rightarrow s = t$

„sections agree locally $\Rightarrow$ agree globally"

2) $U = \bigcup U_i$ open cover, $s_i \in R(U_i)$
$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}\ \forall i,j \Rightarrow \exists s \in R(U)$: $s|_{U_i} = s_i\ \forall i$.

„sections agreeing on overlaps can be glued"

Ex. 1) the presheaf of C^∞ functions on a smooth manifold is a sheaf, same for holomorphic functions.

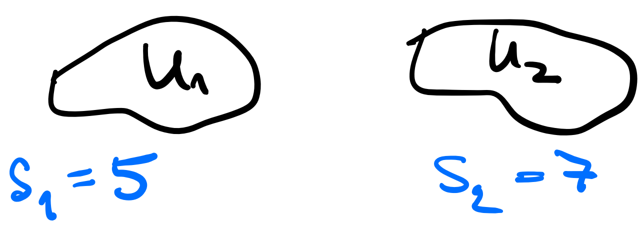
2) continuous k-valued functions form a sheaf of k-algebras:

$$R_X(U) := \left\{ \begin{array}{l} f : U \to \mathbb{A}^1_k \text{ continuous,} \\ \text{i.e. } \forall a \in k \ \ f^{-1}(a) \subseteq U \text{ is closed} \end{array} \right\}$$

- presheaf of k-algebras on X
- locality: $f : U \to \mathbb{A}^1_k$ continuous s.t. $f|_{U_i} = 0 \ \forall i \Rightarrow f = 0$
- gluing: $f_i : U_i \to \mathbb{A}^1_k$ s.t. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \Rightarrow$ define $f(x) := f_i(x) \ \forall i$ s.t. $x \in U_i$

3) bounded functions do NOT form a sheaf: being bounded is not a local condition, so you can't glue a bounded function in general

4) constant presheaf is NOT a sheaf: say, $X = U_1 \amalg U_2$, $A = \mathbb{Z}$



$s_i \in A_X(U_i)$, $s_1|_{U_1\cap U_2} = s_2|_{U_1\cap U_2}$ because $U_1 \cap U_2 = \emptyset$ (assuming $A_X(\emptyset)=0$)

but $\nexists\, s \in A_X(X) = \mathbb{Z}$: $s|_{U_i} = s_i$, because restriction maps are identities.

To fix this, one defines constant sheaf:

$$A_X(U) := \{\text{locally constant } U \to A\} \overset{U \text{ loc-connected}}{=} \prod_{\underbrace{\pi_0 U}_{\text{connected components}}} A$$

This is an example of a more general canonical procedure „sheafification" (see Exercises).

Now, let's focus on the example of continuous k-valued functions.

Constr. pullback of functions

$\varphi : X \to Y$ continuous map of top. spaces $\rightsquigarrow$ gives a „map" from R_Y to R_X

$$\varphi^* : R_Y(U) \longrightarrow R_X(\varphi^{-1}U) \quad \text{-}k\text{-algebra hom.}$$

$$f \longmapsto \varphi^* f := f \circ \varphi$$

$$\begin{tikzcd}
\varphi^{-1}U \arrow[rr, "\varphi"] \arrow[dr, "\varphi^* f"'] & & U \arrow[dl, "f"] \\
& k &
\end{tikzcd}$$

def. A subsheaf $R' \subset R$ is a subfunctor of R that is a sheaf itself.

def. A (k-)ringed space is a pair (X, R):
- X is a top. space
- $R \subseteq R_X$ is k-algebra subsheaf of the sheaf of continuous k-valued functions on X (e.g. polynomial, C^∞, analytic, ...)

A morphism of ringed spaces $(X, R) \xrightarrow[\varphi]{} (Y, S)$ is a continuous map $\varphi : X \to Y$ s.t. $\varphi^* f \in R(\varphi^{-1} U)$ $\forall f \in S(U)$.

Rem. More generally, any top. space with a sheaf of rings is also a ringed space, and the definition of morphisms is more abstract, but in practice we'll be using these specific kinds of sheaves, so we stick to the more concrete def.

„Sheaf" will usually mean a k-algebra subsheaf of R_X.

Ex. smooth manifold = ringed space (X, R) that is locally isom. to $(\mathbb{R}^n, C^\infty)$

complex manifold = ringed space (X, R) that is locally isom. to $(\mathbb{C}^n, A_{\mathbb{C}^n})$

analytic functions

§ Regular functions

following Milne!

def. X alg. set, $U \subseteq X$ open, $f : U \to k$.

1) We say that f is regular at $p \in U$ if one can write in an open $V \ni p$

$$f = \frac{g}{h} \text{ in } V, \quad g, h \in k[X].$$

In particular $h(p) \neq 0$.

2) A function $f : U \to k$ is regular if it's regular $\forall p \in U$.

Ex. $\varphi : X \to Y$ polynomial map of alg. sets, f regular on $U \subset Y \Rightarrow \varphi^* f$ is regular on $\varphi^{-1} U \subseteq X$.

def. $X \subset \mathbb{A}^n$ closed subset. The structure sheaf $\mathcal{O}_X$ of X is defined by

$$\mathcal{O}_X(U) := \{ f : U \to k \mid f \text{ regular on } U \}.$$

It's a k-algebra sheaf, for the same reason as $\mathcal{R}_X$ (regularity is a local condition).

Prop. Every regular function is continous, so $\mathcal{O}_X \subseteq R_X$ is a k-algebra subsheaf.

Proof: $\forall$ regular f is locally $f = \frac{g}{h}$; $g, h \in k[x]$ so it's enough to check for $\frac{g}{h}$, $h \neq 0$.
Both g, h are continuous (polynomial maps).

Want: $\left(\frac{g}{h}\right)^{-1}(a)$ is closed in $D(h)$, for $a \in k$.

$$\left(\frac{g}{h}\right)^{-1}(a) = \underbrace{Z(g - a \cdot h)}_{\text{closed}} \cap D(h) \subset D(h).$$

§Affine varieties

def. An affine variety over k is a k-ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $(W, \mathcal{O}_W)$ where $W \subseteq \mathbb{A}^n$ is a closed subset and $\mathcal{O}_W$ is the structure sheaf of X.

(Isomorphism of k-ringed spaces (X, R) and (Y, S) is a homeomorphism $\varphi: X \to Y$ s.t. $f \in S(U) \Leftrightarrow \varphi^* f \in R(\varphi^{-1}(U))$)

Rem. Some sources use „variety" for „irreducible variety", those are simpler.

Prop. X affine variety, $Z \subseteq X$ closed subset. Then Z is also an affine variety.

Proof: $(X, \mathcal{O}_X) \xrightarrow{\varphi} (W, \mathcal{O}_W)$ for $W \subseteq \mathbb{A}^n$ alg. set, $\mathcal{O}_W$ structure sheaf.

$\Rightarrow (Z, \mathcal{O}_X|_Z) \cong (\varphi(Z), \mathcal{O}_{\varphi(Z)})$ for $\varphi(Z) \subseteq \mathbb{A}^n$ alg. set.

Prop. X an affine variety, $f \in k[X]$. Then $(D(f), \mathcal{O}_X|_{D(f)})$ is an affine variety.

Proof. Assume $X = Z(\mathfrak{a}) \subseteq \mathbb{A}^n$.

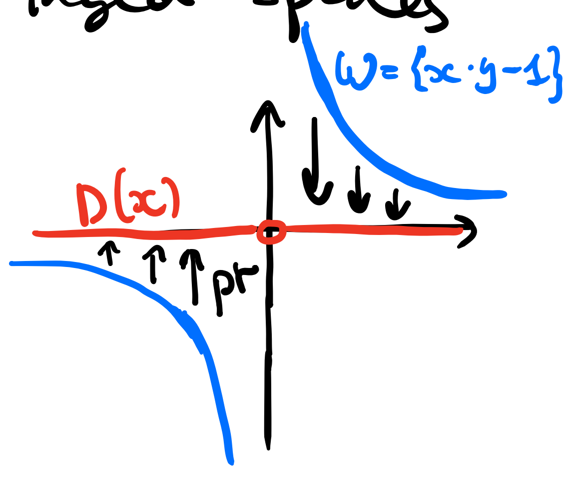
Let $W = Z(\mathfrak{a}, x_{n+1} \cdot f - 1) \subseteq \mathbb{A}^{n+1}$.

We have $pr: W \to X$ (forget x_{n+1}).

$$\begin{tikzcd} W \arrow[r] \arrow[dr, dashed] & X \arrow[d, phantom, "\cup"] \\ & D(f) \end{tikzcd}$$

Projection is a polynomial map $\Rightarrow$ we get a map of k-ringed spaces $pr: W \to D(f)$.

Want: get an inverse.



Define $\alpha : D(f) \to W$

$$(x_1, \dots, x_n) \mapsto \left(x_1, \dots, x_n, \frac{1}{f(x_1, \dots, x_n)}\right),$$

α is clearly an inverse of pr.

To show: α is continuous ($\Rightarrow$ homeo).

$$Z(g) \subseteq W \quad \rightsquigarrow$$

$$\alpha^{-1}(Z(g)) = \{(x_1, \dots, x_n) \in D(f) \mid \underbrace{g\left(x_1, \dots, x_n, \frac{1}{f(x_1, \dots, x_n)}\right)}_{\shortparallel} = 0\}$$

$$0 = \frac{g'(x_1, \dots, x_n)}{f^m(x_1, \dots, x_n)}$$

$$\Updownarrow$$

$$g'(x_1, \dots, x_n) = 0$$

Hence $\alpha^{-1}(Z(g)) = Z(g') \cap D(f) \subset D(f)$ closed.

Remains to check: α is a map of k-ringed spaces, i.e.

$$\varphi \in \mathcal{O}_W(U) \Rightarrow \alpha^* \varphi \in \mathcal{O}_{D(f)}(\alpha^{-1} U).$$

Being regular is local $\Rightarrow$ can assume $\varphi = \frac{s}{t}$ $\Rightarrow$

$$\alpha^* \varphi = \frac{s\left(x_1, \dots, x_n, \frac{1}{f(x_1, \dots, x_n)}\right)}{t\left(x_1, \dots, x_n, \frac{1}{f(x_1, \dots, x_n)}\right)} \overset{\text{clear the denominators}}{=} \frac{s'(x_1, \dots, x_n)}{t'(x_1, \dots, x_n)}$$

Lemma. X Noetherian space (e.g. affine variety), $U \subseteq X$ open. Then U is compact.

Proof. $U = \bigcup_{\alpha \in A} U_\alpha$ infinite open cover; U_α open in X. If there's no finite subcover, you can pick $U_0, U_1, \dots$ s.t. $U_n \not\subseteq V_{n-1} = \bigcup_{i=1}^{n-1} U_i$, then $\{V_0 \subsetneq V_1 \subsetneq \dots\}$ – open chain that doesn't stabilize. ↯

def. An (algebraic) variety $(X, \mathcal{O}_X)$ is a ringed space which is locally isom to an affine variety and quasicompact (compact but not Hausdorff), i.e.

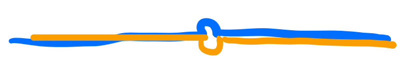
finite $\to$ n

$$X = \bigcup_{i=1}^{n} U_i \quad \text{s.t.} \quad (U_i, \mathcal{O}_X|_{U_i}) \text{ are}$$

isomorphic to (algebraic set, structure sheaf)

We call
- $\mathcal{O}_X$: structure sheaf of X
- sections of $\mathcal{O}_X$: regular functions on X
- morphisms of varieties: regular maps

Rem. Some sources require that a variety has an extra property: it's „separated", which is an analogue of „Hausdorff" condition for manifolds:



is not considered a variety

line with two origins

All the varieties we consider will be separated.

Prop. $(X, \mathcal{O}_X)$ alg variety, $U \subseteq X$ open. Then $(U, \mathcal{O}_U)$ is an algebraic variety.

Proof. Let $X = \bigcup_{i=1}^{n} V_i$ s.t. $(V_i, \mathcal{O}_X|_{V_i})$ are affine varieties. Suffices to show: $(U \cap V_i, \mathcal{O}_{U \cap V_i})$ is an alg variety $\forall i$. Hence we can assume X affine (usual trick!). Then, by the Lemma, U is quasicompact and $\{D(g)\}_{g \in k[X]}$ form a basis $\Rightarrow U = \bigcup_{i=1}^{m} D(g_i)$. Then we win because we've already proved that $(D(g), \mathcal{O}_{D(g)})$ is an affine variety. □

Thm. $(X, \mathcal{O}_X)$ alg variety, $Z \subseteq X$ closed subset. Then $\exists\ \mathcal{O}_Z \subseteq \mathcal{R}_Z$ k-algebra subsheaf s.t. $(Z, \mathcal{O}_Z)$ is an algebraic variety and the inclusion $Z \hookrightarrow X$ is a morphism of varieties.

choice of a structure, not a property!

Proof idea: $\forall\, U \subseteq Z$ open, define

$$\mathcal{O}_Z(U) := \left\{ \begin{array}{l} \text{continuous functions } U \to k \\ \text{that are locally the restriction} \\ \text{of a regular function on } X \end{array} \right\} \subseteq \mathcal{R}_Z(U).$$

Then $\mathcal{O}_Z \subseteq \mathcal{R}_Z$ is a k-algebra subsheaf and $(Z, \mathcal{O}_Z) \hookrightarrow (X, \mathcal{O}_X)$ is a map of k-ringed spaces.

To check: $(Z, \mathcal{O}_Z)$ is a variety (see exercises!)

§ Regular functions on open sets

Recall from CA: R ring, $f \neq 0 \in R$

Localization R_f is the universal R-algebra where f is invertible:

$$R_f = \left\{ \frac{a}{f^m}, \begin{array}{l} a \in R \\ m \in \mathbb{N} \end{array} \right\} \Big/ \sim, \quad \text{where}$$

$$\frac{a}{f^m} \sim \frac{b}{f^n} \quad \text{if} \quad a f^{n+k} = b f^{m+k} \quad (\text{enough } k=1)$$

Thm. X affine var., $h \in k[X]$. Then

$$k[X]_h \longrightarrow \mathcal{O}_X(D(h)) \text{ is a } k\text{-algebra isom}$$

localization

$$\frac{g}{h^m} \longmapsto \left(p \mapsto \frac{g(p)}{h^m(p)} \right)$$

In particular, $h = 1$ gives

$$\boxed{k[X] \simeq \mathcal{O}_X(X)}$$

so regular functions are the same as polynomial functions on X.

Rem. 1) For a general $U \subseteq X$ open, there is no explicit description of $\mathcal{O}_X(U)$

2) Nonetheless the Theorem describes $\mathcal{O}_X$ uniquely as a sheaf of k-valued functions.

Proof: • well-defined and injective: follows from

Claim: $\frac{g}{h^m} = 0$ in $D(h)$ iff $gh = 0$ in $k[X]$

• surjective: fix some $f \in \mathcal{O}_X(D(h))$.
By definition, $\exists$ open cover $D(h) = \bigcup_i V_i$
s.t. $f|_{V_i} = \frac{g_i}{h_i}$; $g_i, h_i \in k[X]$.

Step 1: can assume $V_i = D(h_i)$

We can always assume $V_i = D(a_i)$, $a_i \in k[X]$.
Since $D(a_i) \subset D(h_i)$, we have
$$a_i^{N_i} = h_i \cdot g_i' \text{ for some } N_i \in \mathbb{N},\ g_i' \in k[X].$$
Hence on $D(a_i)$: $f = \frac{g_i}{h_i} = \frac{g_i \cdot g_i'}{h_i \cdot g_i'} = \frac{g_i \cdot g_i'}{a_i^{N_i}}$.

So, after replacing g_i with $g_i \cdot g_i'$ and h_i with $a_i^{N_i}$, we can assume $V_i = D(h_i)$

Step 2: construct the candidate in $k[X]_h$

We have $D(h) = \bigcup_{i \in I} D(h_i)$ and $f|_{D(h_i)} = \frac{g_i}{h_i}$.
Since $D(h)$ is quasicompact, can assume $I = \{1, \dots, m\}$.

• On $D(h_i) \cap D(h_j)$ we have $\frac{g_i}{h_i} = \frac{g_j}{h_j}$ $\Rightarrow$
$$h_i h_j (g_i h_j - h_i \cdot g_j) = 0 \text{ in } k[X]$$
$$\Rightarrow \quad h_i h_j^2 g_i = h_i^2 h_j g_j \quad (*).$$

- Since $D(h) = \bigcup D(h_i) = \bigcup D(h_i^2)$,
  trick
  we have $Z(h) = Z(h_1^2, \dots, h_m^2)$, i.e.

$\exists b_i \in k[X], N \in \mathbb{N}$: $h^N = \sum_{i=1}^{m} b_i h_i^2 \quad (**)$

- Candidate function: $\dfrac{\sum b_i g_i h_i}{h^N} \in k[X]_h$.

Step 3 check that on $D(h)$ we have

$$f = \frac{\sum_{i=1}^{m} b_i g_i h_i}{h^N}.$$

Pick $p \in D(h)$, then $\exists j$: $p \in D(h_j)$. We have:

$$h_j^2 \cdot \sum_i b_i g_i h_i \underset{(*)}{=} \sum_i b_i g_j h_i^2 h_j \underset{(**)}{=} g_j \cdot h_j \cdot h^N \text{ in } k[X].$$

But $f|_{D(h_j)} = \dfrac{g_j}{h_j} \Rightarrow f h_j = g_j$ on $D(h_j)$.

Hence on $D(h_j)$ we get:

$$\cancel{h_j^2} \cdot \sum b_i \cdot g_i \cdot h_i = f \cdot \cancel{h_j^2} \cdot h^N.$$

So, on $D(h_j) \subset D(h)$ we indeed get

$$f = \frac{\sum b_i \cdot g_i \cdot h_i}{h^N}. \qquad \blacksquare$$

Non-example. $U := \mathbb{A}^2 - \{(0,0)\}$ — open subset of $\mathbb{A}^2$, but it's not $D(h)$ because $\dim\{(0,0)\} = 0 \neq \dim Z(h)$.

Claim: $U \overset{j}{\hookrightarrow} \mathbb{A}^2$ induces an isomorphism:

$$k[x,y] \cong \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \xrightarrow[j^*]{\sim} \mathcal{O}_{\mathbb{A}^2}(U).$$

Rem. • This means that for good varieties (called "normal") we can extend a regular function to a codim 2 subset, such as a point in $\mathbb{A}^2$.

• This is similar to Hartogs' phenomenon for holomorphic functions: e.g. an isolated singularity is removable for an analytic function of $n > 1$ variables, but it's not removable for $n = 1$:

e.g. $f(z) = z^{-1}$ is holomorphic in $\mathbb{C} - \{0\}$ but cannot be continued holomorphically to $\mathbb{C}$.

Proof: SECRET (exercise :))

Hint: f regular on $U \Rightarrow$ f regular on $D(x)$ and $D(y)$ simultaneously

§ Stalks

<u>def.</u> Let $(X, \mathcal{O}_X)$ be a k-ringed space. A germ of a function at $p \in X$ is an equivalence class of pairs (U, f) where $U \overset{\neq \emptyset}{\ni} p$ is an open nbhd and $f \in \mathcal{O}_X(U)$; two pairs (U, f) and (U', f') are equivalent if $\exists$ open nbhd $p \in V^{\neq\emptyset} \subset U \cap U'$: $f|_V = f'|_V$.

The stalk of $\mathcal{O}_X$ at p is

$$\mathcal{O}_{X,p} := \{\text{germs of functions at } p\}$$

$(=$ colimit / direct limit of $\mathcal{O}_X(U)$ over $U \ni p\}$

Prop. (properties of stalks)

1) Any stalk $\mathcal{O}_{X,p}$ is a k-algebra:
$(U, f) \dot{+} (U', f') := (U \cap U', f \dot{+} f')$.

2) $U \subseteq X$ open, $p \in U \Rightarrow$
$\mathcal{O}_{U,x} \xrightarrow{\sim} \mathcal{O}_{X,x}$ (inclusion and restriction are inverse isoms)

3) X affine, $p \in X \Rightarrow$

$$\mathcal{O}_{X,p} \simeq k[X]_{m_p}$$

$\leftarrow$ local ring with maximal ideal m_p

where $m_p = \{g \in k[X] \mid g(p) = 0\}$. $\mathcal{O}_{X,p}$ is called the local ring at p. By 2), this makes sense $\forall$ variety X.

Proof of 3): $\mathcal{O}_{X,p} = \{(U \ni p,\ f \in \mathcal{O}_X(U))\}/\sim \ \overset{\text{basis}}{\simeq}$

$\simeq \{(D(h) \ni p,\ f \in \mathcal{O}_X(D(h)))\}/\sim \ \overset{\text{Theorem}}{\simeq}$

$\simeq \{(D(h) \ni p,\ f \in k[X]_h)\}/\sim \ \overset{\text{def.}}{\simeq}$

$\simeq \{f \in k[X]_h \text{ s.t. } h \notin m_p\}/\sim \ \underset{\text{property of localization}}{\simeq} k[X]_{m_p}$.

For X affine, $p \in X$ we get from ③:

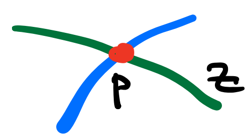
$$\begin{array}{c} \text{prime ideals} \\ \text{of } \mathcal{O}_{X,p} \end{array} \longleftrightarrow \begin{array}{c} \text{prime ideals} \\ \text{of } k[X] \text{ contained in } m_p. \end{array}$$

This means, there is a bijection:

$$\begin{array}{c} \text{prime ideals} \\ \text{of } \mathcal{O}_{X,p} \end{array} \longleftrightarrow \begin{array}{c} \text{irreducible closed} \\ \text{subsets of } X \\ \text{passing through } p \end{array}$$

$\mathfrak{p} = \{(U, f) \mid f|_{U \cap Z} = 0\} \longleftarrow\!\shortmid \ Z \ni p$ irred. closed subset

equivalence classes



minimal prime ideals $\longleftrightarrow$ irreducible components

So, $\mathcal{O}_{X,p}$ is an integral domain iff p lies in a single irreducible component.

Construction. $\varphi:(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ map of k-ringed spaces.

For any $U\subseteq Y$ open we have
$$\varphi^*:\mathcal{O}_Y(U)\to\mathcal{O}_X(\varphi^{-1}(U)).$$

In particular, $\forall p\in X$ it induces
$$\varphi^*:\mathcal{O}_{Y,\varphi(p)}\to\mathcal{O}_{X,p}.$$
$$[(U,f)]\mapsto[(\varphi^{-1}(U),f\circ\varphi)].$$

X,Y varieties $\Rightarrow$ φ^* is a homomorphism of local rings.

By construction, φ^* is a local homomorphism:
$$\varphi^*(m_{\varphi(p)})=m_p.$$

Slogan: life is easy if you are irreducible!

Recall: X irreducible alg set $\Leftrightarrow I(X)$ prime $\Leftrightarrow k[X]$ is an integral domain

def. X irreducible alg. set.
The function field of X is
$$k(X):=\mathrm{Frac}(k[X])$$
localization wrt $k[X]^\times$.

An element $f \in k(X)$ is called a rational function on X.

Every rational function defines a function on some dense open subset of X: $f = \frac{g}{h}$ defines a function on $D(h)$.

Claim: for irreducible X, all the rings attached to X that we discussed, are subrings of $k(X)$. For example:

- $\mathcal{O}_{X,p} = \{ \frac{g}{h} \in k(X) \mid h(p) \neq 0 \}$
- $\mathcal{O}_X(D(h)) = \{ \frac{g}{h^n} \in k(X) \mid g \in k[X],\ n \in \mathbb{N} \}$
- $\mathcal{O}_X(U) = \bigcap_{p \in U} \mathcal{O}_{X,p}$.

§ Morphisms into affine varieties

Lemma. „being a morphism is a local property"
Let X, Y alg varieties, $\varphi : X \to Y$ continuous
Assume there are open covers $X = \bigcup U_i$ and $Y = \bigcup_i V_i$ s.t. $\varphi(U_i) \subseteq V_i$ and $\varphi|_{U_i} : U_i \to V_i$ is a morphism $\forall i$.
Then φ is a morphism.

Reap the benefits of last time work!

Let X be an algebraic variety and $(f_1,\dots,f_m)$ regular functions on X.

This gives a continuous map
$$\varphi: X \to \mathbb{A}^m \qquad x \mapsto (f_1(x),\dots,f_m(x))$$

Claim: φ is a map of k-ringed spaces.

to show: $\forall f \in \mathcal{O}_{\mathbb{A}^m}(U) \qquad \varphi^* f \in \mathcal{O}_X(\varphi^{-1}U)$

By the Lemma, we can assume $U = D(g)$, and so $f = \frac{a}{g^r}, r \geq 0$ (Theorem from last time!)

Then $\varphi^* f = f \circ \varphi = \dfrac{a(f_1,\dots,f_m)}{g^r(f_1,\dots,f_m)}$,

which is regular on $\varphi^{-1}D(g)$ since $g(f_1(x),\dots,f_m(x)) \neq 0$.

Moreover, all morphisms $\varphi: X \to \mathbb{A}^m$ arise this way:
define $f_i := pr_i \circ \varphi$ and get
m regular functions
$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[rr, bend right, "f_i"'] & \mathbb{A}^m \arrow[r, "pr_i"] & \mathbb{A}^1 \end{tikzcd},$$
which give back φ via the construction above.

This reasoning implies the following.

Prop. X alg variety $\Rightarrow$ there's a bijection

$$\left\{ \begin{array}{c} \text{morphisms} \\ \varphi : X \to \mathbb{A}^m \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} k\text{-algebra homs} \\ \varphi^* : k[y_1, \dots, y_m] \to \mathcal{O}_X(X) \\ y_i \longmapsto f_i \end{array} \right\}$$

Moreover, we can generalize this observation.

Prop. X variety Y affine variety $\Rightarrow$

$$\left\{ \begin{array}{c} \text{morphisms} \\ \varphi : X \to Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} k\text{-algebra homs} \\ \varphi^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) \end{array} \right\}$$

Proof: assume $Y \subseteq \mathbb{A}^m$ closed subset, then $\varphi : X \to Y$ is given by

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[dr, dashed] & \mathbb{A}^m \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] \\ & Y \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} \mathcal{O}_X(X) & k[y_1, \dots, y_m] \arrow[l, "\Phi^*"'] \arrow[d] \\ & k[y_1, \dots, y_m]/I(Y) \arrow[ul, dashed] \end{tikzcd}$$

$$\updownarrow$$

$$\mathcal{O}_X(X) \xleftarrow{\varphi^*} \mathcal{O}_Y(Y)$$

k-algebra hom

As a corollary, we get:

Thm (Main theorem of affine varieties)

There is a contravariant equivalence between the category of affine k-varieties and the category of reduced fin. gen. k-algebras:

$$\mathrm{Aff}_k^{op} \xrightarrow{\sim} \mathrm{RedFinGenAlg}_k$$

$$X \longmapsto k[X]$$

$$\varphi : X \to Y \longmapsto \varphi^* : k[Y] \to k[X].$$

This means that the additional condition on φ^* is automatic for morphisms of affine varieties, and that all the topology of affine varieties is encoded in their algebra of regular functions.

Cor. X, Y affine, then
$X \underset{\varphi}{\cong} Y$ iff $k[X] \underset{\varphi^*}{\cong} k[Y]$.

In particular, $k[X]$ does not depend on the choice of embedding $X \subseteq \mathbb{A}^n$.

Ex. 1) $\mathbb{A}^n - 0$, $n \geq 2$

It is NOT an affine variety,
because $\mathbb{A}^n - 0 \overset{j}{\hookrightarrow} \mathbb{A}^n$ is not surjective, hence not an isomorphism of varieties, however j^* is an isomorphism on coordinate rings.

2) twisted cubic

$$\varphi : \mathbb{A}^1 \to C = Z(y - x^2, z - x^3) \subseteq \mathbb{A}^3$$
$$t \mapsto (t, t^2, t^3)$$

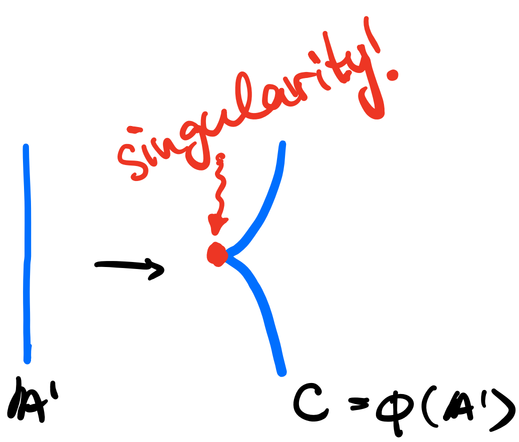
is an isomorphism, because

$$k[x,y,z]/(y-x^2, z-x^3) \xrightarrow{\sim} k[t]$$
$$x \mapsto t$$
$$y \mapsto t^2$$
$$z \mapsto t^3$$

is an isom of k-algebras.

3) cusp

$$\varphi : \mathbb{A}^1 \to \mathbb{A}^2$$
$$t \mapsto (t^2, t^3)$$



$\varphi : \mathbb{A}^1 \to C$ is a bijection but not an isomorphism!

Indeed, on global sections we have

$$k[t] \xleftarrow{\varphi^*} k[x,y]/y^2-x^3$$

$$t^2 \longleftarrow\!\shortmid\; x$$

$$t^3 \longleftarrow\!\shortmid\; y$$

So φ^* is injective but not surjective.

§ Products of varieties

<u>Recall</u>: $X, Y \in \mathrm{Ob}\,\mathcal{C}$ for a category $\mathcal{C}$, then their <u>product</u>, if exists, is an object $X \times Y$ with given pr_X and pr_Y such that $\forall Z \in \mathrm{Ob}\,\mathcal{C}$, $\varphi_X : Z \to X$, $\varphi_Y : Z \to Y$ $\exists!\ \psi : Z \to X \times Y$ that makes the diagram commute:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\varphi_Y"] \arrow[ddr, bend right, "\varphi_X"'] & & \\
& X \times Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"] & Y \\
& X &
\end{tikzcd}$$

universal property of a product

If $X \times Y$ exists, it's unique up to isom.

<u>Thm</u>: 1) The category of affine varieties Aff_k has products, and

$$\boxed{k[X \times Y] \simeq k[X] \otimes_k k[Y]}$$

2) The category of varieties Var_k also has products, obtained by gluing products of affine varieties (we skip the proof of 2))

Proof for affines:

assume $X \subseteq \mathbb{A}^n$, $\quad I(X) = (f_1, \dots, f_r)$

$\qquad\quad Y \subseteq \mathbb{A}^m$, $\quad I(Y) = (g_1, \dots, g_s)$.

Let $I := (f_1, \dots, f_r, g_1, \dots, g_s) \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$,

Facts: 1) $I(X), I(Y)$ radical $\Rightarrow I$ radical

uses $k = \bar{k}$! 2) if $I(X), I(Y)$ are prime, then I is prime

Define $W := Z(I) \subseteq \mathbb{A}^{n+m}$,

$$k[W] \cong k[x_1, \dots, x_n, y_1, \dots, y_m]/I \underset{\text{CA}}{\cong} k[X] \otimes_k k[Y].$$

Projections:

$$\begin{tikzcd} \mathbb{A}^{n+m} \arrow[r] \arrow[d] & \mathbb{A}^m \\ \mathbb{A}^n & \end{tikzcd} \quad \underset{\text{restrict to } W}{\rightsquigarrow} \quad \begin{tikzcd} W \arrow[r, "pr_Y"] \arrow[d, "pr_X"] & Y \\ X & \end{tikzcd}$$

Universal property: Z affine,

$\varphi_X : Z \to X$ and $\varphi_Y : Z \to Y$ polynomial maps,

define $\varphi : Z \to \mathbb{A}^{n+m}$ $\quad z \mapsto (\varphi_X(z), \varphi_Y(z))$.

By construction, $\varphi(Z) = W$ and the diagram commutes:

$pr_X \circ \varphi = \varphi_X$ and $pr_Y \circ \varphi = \varphi_Y$.

Rem. As a set, $X \times Y$ is the Cartesian product, but it has a different topology!

Ex. $\mathbb{A}^n \cong \mathbb{A}^1 \times \dots \times \mathbb{A}^1$ where $\times$ is product in Aff_k.