Chapter 4
Projective varieties

§ Projective space

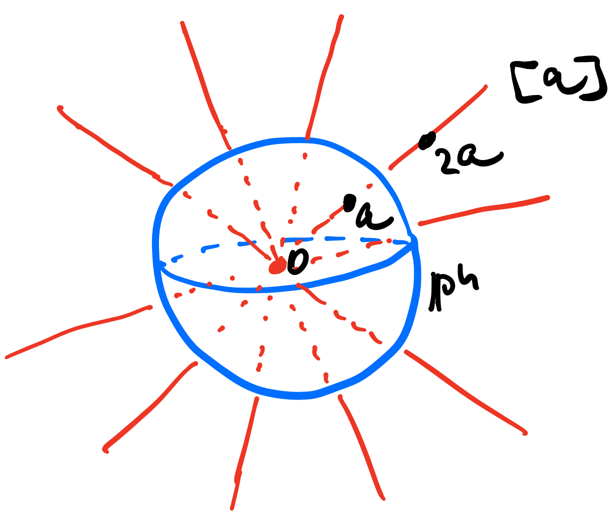
def. The n-dimensional projective space is

$$\mathbb{P}^n := \mathbb{A}^{n+1} - 0 \,/\!\sim, \quad (x_0,\dots,x_n) \sim (\lambda x_0,\dots,\lambda x_n) \quad \forall \lambda \in k^\times$$

$\mathbb{P}^n$ has quotient topology:
$V \subseteq \mathbb{P}^n$ is closed iff $\pi^{-1}(V)$ is closed
for $\pi: \mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$ the projection.
We denote $(x_0 : \dots : x_n) := \pi(x_0, \dots, x_n)$,
these are homogeneous coordinates on $\mathbb{P}^n$
(not unique; beware that $(0:\dots:0)$ is forbidden!)
E.g.: $(1:3) = (2:6)$; $(0:1) = (0:5)$ in $\mathbb{P}^1$.

Intuition:

- Points of $\mathbb{P}^n$ correspond to lines through the origin



• $\mathbb{P}^0 = pt$, $\mathbb{P}^1$ = lines through 0 in $\mathbb{A}^2 \cong \mathbb{A}^1 \cup pt$ ← as sets

More generally, $\mathbb{P}^n$ has an open cover by $n+1$ open sets $\{x_i \neq 0\}$ ($\Leftrightarrow x_i = 1$), which are isomorphic to $\mathbb{A}^n$, and one can choose any of them to decompose

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$$

as sets! ← „hyperplane at ∞"
(depends on the choice of coords!)

In that sense, one can think of $\mathbb{P}^n$ as „compactification" of $\mathbb{A}^n$ (you glue in the „holes at ∞")

§ Projective algebraic sets

def. A graded ring is $R = \bigoplus_{i \geq 0} R_i$ s.t. $f \in R_i$, $g \in R_j \Rightarrow f \cdot g \in R_{i+j}$.

For $f \in R$ we can write $f = f_0 + f_1 + \ldots$

homogeneous components

Ex. $R = k[x_0, .., x_n]$ with
R_d = sums of degree d monomials,
i.e. f s.t. $f(tx_0, .., tx_n) = t^d \cdot f(x_0, .., x_n)$ — homogeneous polynomials of degree d

def. An ideal $I \subseteq R$ is homogeneous if it is generated by homogeneous elements.

Lemma. I, J homogeneous $\Rightarrow$ $I \cap J$, $I + J$, IJ, $\sqrt{I}$ are homogeneous.

def. $f \in k[x_0, \ldots, \hat{x}_i, \ldots, x_n]$ deg d polynomial $\rightsquigarrow$ homogenization of f wrt x_i is

$$f^h(x_0, \ldots, x_n) := x_i^d f\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

homogeneous of deg d

We can recover f from f^h:

$$f = f^h\big|_{x_i = 1}.$$

Ex. $f = x_0^2 + x_1^3 - x_1 \underset{\text{wrt } x_2}{\rightsquigarrow} f^h = x_0^2 x_2 + x_1^3 - x_1 x_2^2$

def. $I = (F_1, \ldots, F_r) \subset k[x_0, \ldots, x_n]$ homogeneous. Then the zero locus of I

$$Z_+(I) := \{(x_0 : \ldots : x_n) \in \mathbb{P}^n \mid F_i(x_0, \ldots, x_n) = 0 \ \forall i\}$$

is a closed subset of $\mathbb{P}^n$, because $\pi^{-1}(Z_+(I)) = Z(I) \cap (\mathbb{A}^{n+1} - 0)$.

We call $Z_+(I)$ projective algebraic sets.

Lemma. They satisfy similar properties:

1) $Z_+(\mathfrak{a}\cdot\mathfrak{b}) = Z_+(\mathfrak{a}) \cup Z_+(\mathfrak{b})$
2) $Z_+(\mathfrak{a}+\mathfrak{b}) = Z_+(\mathfrak{a}) \cap Z_+(\mathfrak{b})$
3) $Z_+(\mathfrak{a}) = Z_+(\sqrt{\mathfrak{a}})$

Conversely, for any $X \subseteq \mathbb{P}^n$ define

$$I(X) := \left(f \in k[x_0,\dots,x_n] \;\middle|\; \begin{array}{l} f \text{ homogeneous} \\ f(x)=0 \ \forall x \in X \end{array} \right)$$

def. An algebraic set $W \subseteq \mathbb{A}^{n+1}$ is a **cone** if $0 \in W$ and $x \in W \Rightarrow \lambda \cdot x \in W$ $\forall \lambda \in k$.

- For a cone W, its **projectivization** is $\mathbb{P}(W) := \pi(W - 0) \subseteq \mathbb{P}^n$
- For a closed subset $X \subseteq \mathbb{P}^n$, its **cone** is

$$C(X) := \pi^{-1}(X) \cup \{0\} \subseteq \mathbb{A}^{n+1}.$$

Cones are exactly zero loci of homog. ideals.

Prop. There is a bijection

$$\{\text{cones in } \mathbb{A}^{n+1}\} \leftrightarrow \{\text{projective algebraic sets}\}$$

$$C(X) = Z(\mathfrak{a}) \qquad\qquad X = Z_+(\mathfrak{a})$$

Proof: $\mathfrak{a} \subset k[x_0,\dots,x_n]$ homogeneous $\Rightarrow$ $\mathbb{P}(Z(\mathfrak{a})) = Z_+(\mathfrak{a})$ and $C(Z_+(\mathfrak{a})) = Z(\mathfrak{a})$, and any cone is of this form.

def. For a graded ring $\bigoplus_{d\geq 0} R_d$ its <u>irrelevant ideal</u> is $\bigoplus_{d>0} R_d$.

For example, $Z_+((x_0,\dots,x_n)) = \emptyset$.

Thm (Projective Nullstellensatz)

1) $\forall$ projective alg set $X \subset \mathbb{P}^n$ we have
$$Z_+(I(X)) = X$$

2) $\forall$ homogeneous ideal $J \subset k[x_0,\dots,x_n]$ with $\sqrt{J} \neq (x_0,\dots,x_n)$, we have
$$I(Z_+(J)) = \sqrt{J}.$$

This gives an inclusion-reversing bijection:

$$\left\{\begin{matrix}\text{projective} \\ \text{alg sets in } \mathbb{P}^n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{homogeneous radical} \\ \text{ideals in } k[x_0,\dots,x_n] \\ \neq (x_0,\dots,x_n)\end{matrix}\right\}$$

$$X \longmapsto I(X)$$
$$Z_+(J) \longleftarrow J$$

Proof of 2):

$$I(Z_+(J)) = \left(\text{homog. } f \in k[x_0, \dots, x_n] \;\middle|\; f(x) = 0 \ \forall x \in Z_+(J)\right)$$

$$= \left(-\|- \;\middle|\; f(x) = 0 \ \forall x \in Z(J) - 0 \right) =$$

$$\underset{\text{zero locus is closed}}{=} \left(-\|- \;\middle|\; f(x) = 0 \ \forall x \in \underbrace{\overline{Z(J) - 0}}_{= Z(J)} \right) =$$

homog. polynomials vanish at 0

$$= \left(-\|- \;\middle|\; f(x) = 0 \ \forall x \in Z(J) \right) \underset{\substack{Z(J) \text{ is a cone,} \\ \text{so its ideal is necessarily homogeneous}}}{=} I(Z(J)) =$$

$$= I(Z(J)) \underset{\text{usual Nullstellensatz}}{=} \sqrt{J}.$$

§ Projective varieties

def. The distinguished open sets defined by x_i are

$$D_+(x_i) = \{(x_0 : \dots : x_n) \mid x_i \neq 0\} \subset \mathbb{P}^n$$
$$\parallel$$
$$\{(x_0 : \dots : 1 : \dots : x_n)\}$$

Consider $A_i := Z(x_i - 1) \subset \mathbb{A}^{n+1} - 0$,
then $A_i \simeq \mathbb{A}^n$ and the map

$\pi|_{A_i} : A_i \to D_+(x_i)$ is a bijection,

with inverse $\alpha : (x_0 : \dots : x_n) \mapsto \left(\frac{x_0}{x_i}, \dots, 1, \dots, \frac{x_n}{x_i}\right) \in A_i$.

Will show: $\{D_+(x_i)\}$ is an affine cover that makes $\mathbb{P}^n$ into a variety.

Ex. conics in $\mathbb{P}^2$

projective conic $xy = z^2$

$\to$ in $D_+(z)$: hyperbola

$\to$ in $D_+(x)$: parabola

So in fact, all conics are up to the choice of coords the same when considered as a conic in $\mathbb{P}^2_k$, $k = \bar{k}$.

Construction: $g, h \in k[x_0, .., x_n]_d$ - homogeneous of deg d

Then $\dfrac{g(tx_0, .., tx_n)}{h(tx_0, .., tx_n)} = \dfrac{\cancel{t^d}\, g(x_0, .., x_n)}{\cancel{t^d}\, h(x_0, \dots, x_n)}$,

so $\frac{g}{h}$ gives a rational function on $\mathbb{P}^n$.

def. $X \subseteq \mathbb{P}^n$ projective alg set. Its structure sheaf is

$$\mathcal{O}_X(U) := \left\{ f : U \to k \;\middle|\; \begin{array}{l} f \text{ is continuous and locally of the form} \\ \dfrac{g(x_0, .., x_n)}{h(x_0, \dots, x_n)} \text{ homogeneous of degree } d \end{array} \right\}$$

$\mathcal{O}_X$ is a sheaf of k-algebras.

Now, $\mathbb{P}^n = \bigcup_i D_+(x_i)$ is an affine open cover.

Prop. $\pi|_{A_i} : A_i \to D_+(x_i)$ is an isom of k-ringed spaces.

Proof: • continuous and bijective - ok

• homeo: want $\pi(Z(f) \cap A_i) \subseteq D_+(x_i)$ closed, $f \in k[x_0, .., \hat{x_i}, .., x_n] \cong k[A_i]$

We have: $\pi(Z(f) \cap A_i) = \{(x_0 : \dots : x_n) \mid x_i = 1 \text{ and } f(x_0, .., x_n) = 0\} =$

$= \{(x_0 : \dots : x_n) \mid x_i = 1 \text{ and } f^h(x_0, .., x_n) = 0\} = Z_+(f^h) \cap D_+(x_i)$.

wrt x_i

· homs of k-ringed spaces: similar to the proof that $D(h)$ are affine varieties.

Cor. $X \subseteq \mathbb{P}^n$ proj. alg set $\Rightarrow (X, \mathcal{O}_X)$ is a variety, with affine cover
$$X = \bigcup_{i=0}^{n} X \cap D_+(x_i).$$
We call them projective varieties.

An open subvariety of a projective variety is called a quasiprojective variety.

def. $X \subseteq \mathbb{P}^n$ proj. variety. Its homogeneous coordinate ring is
$$S(X) := k[C(X)],$$
where $C(X) \subseteq \mathbb{A}^{n+1}$ is the cone over X.

The ring $S(X)$ is graded, since $X = Z_+(\mathfrak{a})$ for $\mathfrak{a}$ homogeneous, and $C(Z_+(\mathfrak{a})) = Z(\mathfrak{a})$.

Ex: 1) $S(\mathbb{P}^n) \simeq k[x_0, \dots, x_n]$

2) C projective twisted cubic

$C = \operatorname{Im} \varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^3 \qquad (x_0 : x_1) \mapsto (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3)$

$$S(C) \simeq k[x_0, \underset{x}{\overset{\shortparallel}{x_1}}, \underset{y}{\overset{\shortparallel}{x_2}}, \underset{z}{\overset{\shortparallel}{x_3}}] \Big/ (x_0 x_2 - x_1^2,\ x_1 x_3 - x_2^2,\ x_0 x_3 - x_1 x_2)$$

to make the ideal radical $\rightsquigarrow$

§ Regular functions on projective varieties

Recall: $X \subseteq \mathbb{A}^n$ affine $\Rightarrow \mathcal{O}_X(X) \simeq k[X]$, and $k[X]$ recovers all the info about X.

In contrast, we will prove:

Thm. $X \subseteq \mathbb{P}^n$ connected projective variety. Then

$$\mathcal{O}_X(X) \simeq k$$

Rem: Liouville's thm in complex analysis says $f: \mathbb{CP}^1 \to \mathbb{C}$ holomorphic $\Rightarrow f|_{\mathbb{C}}: \mathbb{C} \to \mathbb{C}$ bounded holom $\Rightarrow f$ is constant.

Prop. $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) \simeq k$.

Proof:

$$\begin{tikzcd} \mathbb{A}^{n+1}-0 \arrow[d, "\pi"'] \arrow[dr, "\pi^* f = f \circ \pi"] & \\ \mathbb{P}^n \arrow[r, "f"'] & k \end{tikzcd}$$

$\Rightarrow \pi^* f$ is regular on $\mathbb{A}^{n+1}-0$

$\Rightarrow \pi^* f$ is a polynomial in $x_0, \dots, x_n$.

But $\pi^* f$ is constant on lines through the origin $\Rightarrow$ it must be constant.

Exercise X irreducible alg set $\Rightarrow$

$k(X) = \mathrm{Frac}(k[X]) \simeq \{(U, f)\}/\sim$ where $U \subseteq X$ dense open, $f \in \mathcal{O}_X(U)$.

Proof of thm. We can assume $X \subseteq \mathbb{P}^n$ irreducible and $X \not\subseteq \{x_i = 0\}$ $\forall i$ (otherwise $n \rightsquigarrow n-1$).

Let $f: X \to k$ be a global regular function.

$$\begin{tikzcd} C(X)-0 \arrow[d, "\pi"] \arrow[dr, "\pi^* f := F \in k(C(X))"] & \\ X \arrow[r, "f"'] & k \end{tikzcd}$$

Let $D_i := C(X) \cap D(x_i)$, each D_i is affine open, non-empty $\Rightarrow$ dense in $C(X)$.

with $k[D_i] \simeq S(X)_{x_i}$ – localization

We have $F\big|_{D_i} = \dfrac{g_i}{x_i^{r_i}}$ for $g_i \in k[C(X)]$.

Since F is constant on lines through 0, g must be homogeneous of deg r_i,

so $g_i \in S(X)_{r_i}$ graded piece and $x_i^{r_i} \cdot F = g_i$ in $D_i - 0 \Rightarrow$ in $C(X) - 0$.

Choose $r > \sum_i r_i$.

Then $\forall$ monomial $x_0^{a_0} \cdot \ldots \cdot x_n^{a_n}$ of deg r

has some $x_i^{a_i} = x_i^{r_i} \cdot x_i^{j}$, $j > 0$.

Hence $x_0^{a_0} \cdot \ldots \cdot x_n^{a_n} \cdot F \in S(X)_r$, so

$\cdot F: S(X) \to S(X)$ leaves the subspace $S(X)_r$ invariant.

Cayley-Hamilton thm $\Rightarrow$ F satisfies

$F^m + b_{m-1} F^{m-1} + \ldots + b_0 = 0$, $b_i \in k$

$\Rightarrow F \in k(C(X))$ is algebraic over k $\underset{k = \bar{k}}{\Longrightarrow}$ $F \in k$. $\blacksquare$

Cor. $f: X \to Y$ morphism $\Rightarrow f$ is constant on connected comps.
projective, affine

Proof: Morphisms $X \to Y$ with Y affine correspond to k-algebra homs
$\mathcal{O}(Y) \to \mathcal{O}_X(X) = k$, so all maps are constant.
for X connected

Cor. X projective & affine connected variety $\Rightarrow X = pt.$

Proof: X affine $\Rightarrow \exists i: X \hookrightarrow \mathbb{A}^n$
X projective $\Rightarrow i$ is constant.

§ A topological detour

We had: $\mathbb{P}^n := \mathbb{A}^n - 0/\sim$ where $x \sim \lambda \cdot x$, $\lambda \in k^\times$

- $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{C}P^n$ are isomorphic as sets but have different topologies:
both topologies are defined so that regular $\mathbb{C}$-valued functions would be continuous, but $\mathbb{C}$ can have Zariski topology ($\mathbb{P}^n_{\mathbb{C}}$) or Euclidean topology ($\mathbb{C}P^n$).

- More generally, there's a functor $\mathrm{Re}: \mathrm{Var}_{\mathbb{C}} \to \mathrm{Top}$ called Betti realization, induced by $\underset{\text{zar. top.}}{\mathbb{A}^1_{\mathbb{C}}} \mapsto \underset{\text{Euclidean top}}{\mathbb{C}}$.

Later we will study smooth vs singular varieties, and the functor restricts to

$$\mathrm{Re}: \mathrm{SmVar}_{\mathbb{C}} \to \mathrm{SmMan}.$$

§ Morphisms of projective varieties

Suppose $\varphi: X \to Y$ is a continuous map of projective varieties, that fits into

$$\begin{tikzcd}
C(X)-0 \arrow[r, "\Phi"] \arrow[d, "\pi"'] & C(Y)-0 \arrow[d, "\pi"] \\
X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

Lemma. Φ is a morphism $\Rightarrow$ so is φ.

Proof:

$A_i := \{x_i = 1\}$

$$\begin{tikzcd}
A_i \cap C(X)-0 \arrow[r, hook] & C(X)-0 \arrow[r, "\Phi"] \arrow[d] & C(Y)-0 \arrow[d] \\
D_+(x_i)\cap X \arrow[u, "s"] \arrow[r, hook] \arrow[rr, bend left] & X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

composition of morphisms!

Ex. ① prototype example

$X \subseteq \mathbb{P}^n$, $f_0, \dots, f_m$ homogeneous of deg d
$Z(f_0, \dots, f_m) \cap X = \emptyset$ $\rightsquigarrow$

$$X \longrightarrow \mathbb{P}^m$$
$$x \longmapsto (f_0(x) : \dots : f_m(x))$$

② projection

We want a map $\mathbb{P}^n \overset{p}{\dashrightarrow} \mathbb{P}^m$ $n > m$
induced from $\mathbb{A}^{n+1} \longrightarrow \mathbb{A}^{m+1}$

$$(x_0, \dots, x_n) \longmapsto (x_0, \dots, x_m)$$

Such a map p is a morphism outside $\mathbb{P}^{n-m-1} \subset \mathbb{P}^n$ given by $\{x_0 = \dots = x_m = 0\}$.
(Later: it is a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$).

Special case: $p : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$
projection from the point $(0 : \dots : 0 : 1) \in \mathbb{P}^n$.

def. A map $\varphi : Y \to X$ is a closed embedding if $\varphi(Y) \subseteq X$ is closed and
$\varphi : Y \to \varphi(Y) \subseteq X$ is an isomorphism.
canonical variety structure

Being a closed embedding is a local property on the image (enough to take $\varphi(Y) = \bigcup U_i$).

Ex. 1) Segre embedding

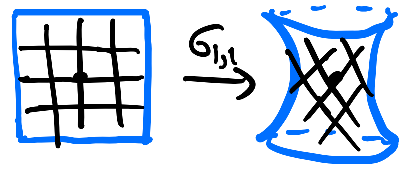
$$\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{(m+1)(n+1)-1}$$

$$(x_0:\ldots:x_m),(y_0:\ldots:y_n) \mapsto (x_0y_0 : x_0y_1 : \ldots : x_my_n)$$

lexicographic order

Fact: it's a closed embedding.

$m=n=1$ $\quad \sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$



$\mathrm{Im}\ \sigma_{1,1} = Z_+(u_0u_3 - u_1u_2) \subset \mathbb{P}^3$

Cor. $X \subseteq \mathbb{P}^m$, $Y \subseteq \mathbb{P}^n$ projective varieties $\Rightarrow$ $X \times Y$ is projective.

Proof: $X \times Y \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is a closed subvariety $\Rightarrow$ $X \times Y \hookrightarrow \mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{mn+m+n}$ is a closed embedding.

2) Veronese embedding

fix $d > 0$ $\rightsquigarrow$ $N := \binom{n+d}{d} - 1$

$$\varphi_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$$

$$(x_0:\ldots:x_n) \mapsto (x_0^{i_0} \cdot \ldots \cdot x_n^{i_n})$$

$$i_0 + \ldots + i_n = d$$

← monomials of deg d in lexicographic order

$n=1$: $\quad \mathbb{P}^1 \longrightarrow \mathbb{P}^d$

$$(x_0:x_1) \mapsto (x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d)$$

„rational normal curve"

§ Grassmannians

(Chapter 13 in Ottem–Ellingsrud, older version)

$Gr(r,n)$ parametrizes r-dimensional linear subspaces $W < k^n$.

$r=1 \rightsquigarrow Gr(1,n+1) = \mathbb{P}^n$

Goal: construct $Gr(r,n)$ as a projective variety and describe its affine charts (open cover).

Main idea: an r-dimensional subspace $W < k^n$ is determined by a basis $w_1,\dots,w_r \in W$, which can be written as an $r \times n$ matrix, but this choice is not unique.

There's an action (left multiplication)

$$GL_r(k) \times M_{r\times n}(k) \to M_{r \times n}(k)$$
$$(A, M) \longmapsto A \cdot M$$

and r-dimensional subspaces correspond to orbits, i.e. we want to take a quotient by the action of $GL_r(k)$.

Similarly to:

$\mathbb{P}^n = \mathbb{A}^{n+1} - 0/_\sim$ = quotient of $\mathbb{A}^{n+1} - 0$ by the scaling action of $k^\times = GL_1(k)$.

We'll assume $r=2$ just to simplify notations.

Construction. Consider $M_{2,n} \simeq \mathbb{A}^{2n}$

$\begin{pmatrix} x_1 \dots x_n \\ y_1 \dots y_n \end{pmatrix} \rightsquigarrow$ coords $(x_1, \dots, x_n, y_1, \dots, y_n)$

Let $U \subset M_{2,n}$ be the matrices of full rank, i.e. at least one of the 2×2-minors is non-zero:

$$U = M_{2,n} - Z(\{\det M_{ij} \mid 1 \leq i < j \leq n\})$$

U open subset of $M_{2,n}$; M_{ij} 2×2 matrix $\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$

Consider
$$\varphi : U \longrightarrow \mathbb{P}^{\binom{n}{2}-1}$$
$$M \mapsto (\det M_{ij})$$

We observe: ① φ is constant on $GL_2(k)$-orbits, i.e. if $M' = A\cdot M$ for $A \in GL_2(k)$ then
$\det M'_{ij} = \underset{\in k^\times}{\det A} \cdot \det M_{ij}$, so $\varphi(M) = \varphi(M')$.

② M is determined by $\varphi(M)$ up to $GL_2(k)$-action, because by applying elementary ~~raw~~ row operations we can make M of the form

$\begin{pmatrix} * \dots 1 & * & \dots & 0 & * & \dots \\ * \dots 0 & * & \dots & 1 & * & \dots \end{pmatrix}$, and the entries $*$ are determined by $\varphi(M)$ because they are $\pm \det M_{ij}$.

This reasoning proves the following.

Prop. There's a natural bijection of sets:

$$\left\{\begin{array}{c}\text{2-dimensional}\\ \text{subspaces}\\ \omega \subset k^n\end{array}\right\} \longleftrightarrow \left\{ U/GL_2(k) \right\}$$

$$\omega \longmapsto M_\omega$$

Moreover, φ induces a continuous injective map

$$\varphi : U/GL_2(k) \longrightarrow \mathbb{P}^{\binom{n}{2}-1}$$

quotient topology: $M \sim M'$ if $M' = A \cdot M$ for $A \in GL_2(k)$

def. The Grassmannian variety is

$$Gr(2,n) := \operatorname{Im} \varphi \subset \mathbb{P}^{\binom{n}{2}-1}$$

Thm. 1) $Gr(2,n) \subset \mathbb{P}^{\binom{n}{2}-1}$ is an irreducible projective variety, defined by equations

(*) $\quad u_{ij} \cdot u_{kl} - u_{ik} \cdot u_{jl} + u_{jk} \cdot u_{il} = 0 \qquad 1 \leq i < j < k < l \leq n$

Plücker relations

2) $Gr(2,n)$ has an open cover by $\binom{n}{2}$ copies of $\mathbb{A}^{2n-4}$.

3) $\varphi : U/GL_2(k) \xrightarrow{\sim} Gr(2,n)$ is a homeomorphism.

Rem. $2 \rightsquigarrow m$: (*) get more complicated, the rest is the same.

Proof: 1) • irreducible: because $U \subset \mathbb{A}^{2n}$ open is irreducible
• closed subset of $\mathbb{P}^{\binom{n}{2}-1}$:

let I be the ideal denoted by $(*)$,
want to show $\mathrm{Im}(\varphi) = Z_+(I)$.

$\subseteq$ follows from

$$\det \begin{pmatrix} x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \\ x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \end{pmatrix} = 0$$

$\supseteq$ take any $u \in Z(I)$, say in the chart $u_{12} = 1$.
Then $i=1, j=2$ gives

$$u_{kl} = u_{1k} u_{2l} - u_{2k} u_{1l},$$

which expresses any u_{kl} in terms of u_{1l} and u_{2k}.
Hence $u = \varphi(M)$ for

$$M = \begin{pmatrix} 1 & 0 & -u_{23} & \cdots & -u_{2n} \\ 0 & 1 & u_{13} & & u_{1n} \end{pmatrix}.$$

2) Let $U_{ij} \subset U$ be the closed subset given by

$$\left\{ \begin{array}{c} \\ \\ \end{array} \begin{array}{cccccccc} & & i & & & j & & \\ \big(* & \cdots & 1 & * & \cdots & 0 & * & \cdots \\ * & \cdots & 0 & * & \cdots & 1 & * & \cdots \big) \end{array} \right\}.$$

Then the map

$$\varphi|_{U_{ij}} : \underset{\substack{\text{Is} \\ \mathbb{A}^{2n-4}}}{U_{ij}} \to D_+(u_{ij}) \cap Gr(2,n) =: D_{ij}$$

is surjective (by row operations argument), and it's an embedding in $D_+(u_{ij})$

because the induced map

$$\underset{\text{emb}}{\varphi|_{U_{ij}}} : \mathbb{A}^{2n-4} \xrightarrow{\subset \cdot\cdot \text{pr.}} \mathbb{A}^{\binom{n}{2}-1} \simeq D_+(u_{ij})$$

has a left inverse (the projection to corresponding minors).

Hence $\varphi|_{U_{ij}} : U_{ij} \to D_{ij}$ is an isom.

We get that

$$Gr(2,n) = \bigcup_{i<j} D_{ij} \simeq \bigcup_{\binom{n}{2}} \mathbb{A}^{2n-4}$$

and these are affine open charts of $Gr(2,n)$.

3) We have:

$$\varphi : U/_{GL_2(k)} \to Gr(2,n)$$

is a continuous bijection.

To show that it's a homeo, enough to restrict φ to $\varphi^{-1}(D_{ij}) \simeq U_{ij}$, and we have proved before that it's an isom.

Ex: $Gr(2,4) \subseteq \mathbb{P}^5$ is a quadric hypersurface, cut out by $u_{12}u_{34} - u_{13}u_{24} + u_{14}u_{23} = 0$.

$U_{12} = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$, $U_{13} = \begin{pmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{pmatrix}$, ... – 6 charts isomorphic to $\mathbb{A}^4$.