Chapter 5
Dimension revisited
§ Motivation

Previously in the series:

- X top. space $\rightsquigarrow$ its **dimension** is

  $\dim X := \sup\{r \mid X_0 \subsetneq \dots \subsetneq X_r$ irreducible closed subsets of $X\}$

- X affine variety $\Rightarrow$ $\dim X = \dim k[X]$

  ↖ Krull dimension

Problem: how to compute dimension of varieties?

Prop X top. space, $X = \bigcup_{i \in I} U_i$ open cover.
Then $\dim X = \sup_{i \in I} \dim U_i$.

Proof: • $\dim X \geq \dim U_i$ — proved before (true $\forall X$)
• $\dim X \leq \sup_i \dim U_i$:

pick a chain $\{Z_0 \subsetneq \dots \subsetneq Z_d\}$ - irreducible subsets of X.
Choose $U := U_i$ s.t. $Z_0 \cap U_i \neq \emptyset$.
Claim: $\{Z_0 \cap U \subset \dots \subset Z_d \cap U\}$ — chain of irreducibles in U.
Indeed, $Z_j \cap U \subset Z_j$ is a non-empty open $\Rightarrow$
it's irreducible and dense. Hence $\overline{Z_j \cap U} = Z_j \subset X$,
closure in X
so $Z_j \cap U \neq Z_{j+1} \cap U$.

Another approach:

- consider each irreducible component separately, and dim is the maximum of dim's of components
- for each component use the following

Fact X irreducible variety, $U \subseteq X$ open $(\neq \emptyset)$. Then $\dim U = \dim X$.

Rem $\leq$: general fact for top. spaces
$\geq$: specific for algebraic varieties

Ex. 1) $\mathbb{P}^n = \mathbb{A}^{n+1}-0/\sim$ is irreducible, $\mathbb{A}^n \subset \mathbb{P}^n$ open $\Rightarrow$ $\dim \mathbb{P}^n = n$

2) $Gr(2,n)$ is irreduble; has an open cover by $D_{ij} \simeq \mathbb{A}^{2n-4}$ $\Rightarrow$

$$\dim Gr(2,n) = 2\cdot(n-2).$$

More generally, $\dim Gr(d,n) = d\cdot(n-d)$.

The Fact follows from the following important result (prove later).

Thm. X irreducible variety, then

$$\dim X = \mathrm{tr}\dim_k k(X)$$

transcendence degree

§ Dominant and finite morphisms

def. A morphism of varieties $\varphi: X \to Y$ is **dominant** if $\varphi(X) \subseteq Y$ is dense.

Ex: 1) $\mathbb{A}^n \underset{\text{chart}}{\hookrightarrow} \mathbb{P}^n$ is dominant

2) $\mathbb{A}^n \underset{\text{hypersurface}}{\hookrightarrow} \mathbb{A}^{n+1}$ is not dominant

Prop. $\varphi: X \to Y$ morphism of affine varieties. Then:
φ dominant $\Leftrightarrow$ $\varphi^*: k[Y] \to k[X]$ is injective.

Proof: ($\Rightarrow$) let $f \in k[Y]$, $f \neq 0$

φ dominant $\Rightarrow \varphi(X) \cap D(f) \neq \emptyset \Rightarrow$
$f \circ \varphi \neq 0$ on $\varphi^{-1}(D(f)) \neq \emptyset \Rightarrow \varphi^* f \neq 0$

($\Leftarrow$) suppose $\overline{\varphi(X)} =: Z \subsetneq Y$

Y affine $\Rightarrow Z = Z(I)$ for a non-zero ideal $I \subset k[Y]$

Pick $f \in I$, $f \neq 0$. We have:
f vanishes along $Z \Rightarrow \varphi^* f = 0$ on X.

def. A morphism of affine varieties $X \xrightarrow{\varphi} Y$ is **finite** if $k[X]$ is a fin. gen. $k[Y]$-module via φ^*.

Exercise: finite $\underset{\not\Leftarrow}{\Rightarrow}$ all fibers are finite

def. Let $\varphi: X \to Y$ a morphism of varieties.

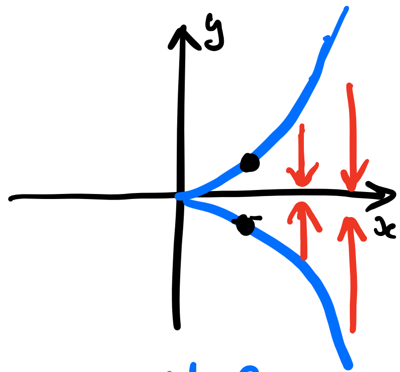
φ is **affine** if $\forall$ affine $V \subseteq Y$ $\varphi^{-1}(V) \subseteq X$ is affine.

φ is **finite** if it's affine and

$\forall y \in Y$ $\exists$ affine $V \ni y$ s.t. $\varphi: \varphi^{-1}V \to V$ is finite.

Ex. 1) $X := Z(x^3 - y^2) \subset \mathbb{A}^2$

$\varphi: X \to \mathbb{A}^1$ projection to x-axis



cuspidal cubic

$\varphi^*: k[\mathbb{A}^1] \to k[X]$

$k[x] \hookrightarrow k[x,y] / y^2 - x^3$

as $k[x]$-module is $k[x] \cdot 1 \oplus k[x] \cdot y$

$\Rightarrow \varphi$ is finite.

rank 2 module, corresponds to most fibers being 2 points

2) $\varphi: \mathbb{A}^1 - 0 \hookrightarrow \mathbb{A}^1$ „open embedding"

$\hookrightarrow \varphi^*: k[t] \hookrightarrow k[t, t^{-1}]$

is NOT finite, although it has finite fibers!

NB: closed embeddings are finite, whereas open embeddings are not.

Lemma. $\varphi: X \to Y$ affine varieties, $x \in X$, $y \in Y$.

Then $\varphi(x) = y$ iff $\varphi^* m_y \subseteq m_x$ in $k[X]$.

functions vanishing at x

Proof: $\varphi^{-1}(y) = \{x \in X \mid \varphi(x) = y\} =$

$= \{x \in X \mid f(\varphi(x)) = 0 \ \ \forall f \in m_y\} = Z(\varphi^* m_y)$.

Prop. $\varphi: X \to Y$ finite morphism of varieties.
Then: 1) φ is closed ← sends closed subsets to closed ones

2) if φ is also dominant, then φ is surjective.

Proof. 1) $\Rightarrow$ 2) but we want to prove 1) using 2).
We can assume X, Y affine.

② φ dominant $\Rightarrow k[Y] \overset{\varphi^*}{\hookrightarrow} k[X]$ (proved above).
If $\exists\, y \in Y$ s.t. $y \notin \varphi(X)$, then

$$\varphi^* m_y \underset{\text{injectivity}}{=} m_y \cdot k[X] \underset{\text{Lemma}}{=} k[X].$$

$k[X]$ is a fin.gen. $k[Y]$-module $\Rightarrow$ by Nakayama lemma
$\exists\, a \in m_y : (1-a) \cdot k[X] = 0.$
But this would imply
$0 = (1-a)\cdot 1 = \varphi^*(1-a) \Rightarrow a = 1$ by injectivity of φ^*
$\Rightarrow 1 \in m_y$ - contradiction!

① assume $Z \subseteq X$ closed
• Z irreducible $\Rightarrow W := \overline{\varphi(Z)} \subset Y$ closed irred.
and $\varphi|_Z : Z \to W$ is dominant
and finite ($k[X]$ generated by $x_1, \dots, x_d$
over $k[Y] \Rightarrow k[Z]$ generated by $\bar{x}_1, \dots, \bar{x}_d$ over $k[W]$).
Hence by ②, $\varphi|_Z$ is surjective $\Rightarrow \varphi(Z) = W$ is closed.
• $Z = Z_1 \cup \dots \cup Z_r$ irred. $\Rightarrow \varphi(Z) = \varphi(Z_1) \cup \dots \cup \varphi(Z_r)$ is closed.

We can refine it:

Prop (Going Up) appeared in Comm Alg!

$\varphi: X \to Y$ finite dominant morphism, $Z \subseteq Y$ closed irreducible subset $\Rightarrow$ $\exists\, W \subseteq X$ closed irreducible s.t. $\varphi(W) = Z$.

Proof: By previous Prop, φ surjective $\Rightarrow$ $\varphi^{-1}(Z) \subseteq X$ closed non-empty $\Rightarrow$

$\varphi^{-1}(Z) = W_1 \cup \dots \cup W_r$ irreducible comps

$\Rightarrow \varphi(W_1) \cup \dots \cup \varphi(W_r) = Z$

$\Rightarrow \exists\, i: \varphi(W_i) = Z$ because Z irred.

Lemma.

$\varphi: X \to Y$ finite dominant morphism, $Z \subset X$ proper closed subset $\Rightarrow \varphi(Z) \subset Y$ is proper.

Proof. We can assume X, Y affine irreducible.

Suppose $\varphi(Z) = Y$, let $f \in I(Z)$.

φ finite $\Rightarrow k[X]$ is a fin.gen. $k[Y]$-module $\Rightarrow$ $\exists$ relation

$$f^n + \varphi^*(a_{n-1}) f^{n-1} + \dots + \varphi^*(a_0) = 0$$

with $a_0, \dots, a_n \in k[Y]$ s.t. n is minimal.

Since $f(x) = 0 \quad \forall x \in Z$,

$\varphi^*(a_0)(x) = 0$ so a_0 vanishes along $\varphi(Z) = Y$, hence $a_0 = 0$ because φ^* injective.

This contradicts n minimal $\Rightarrow f = 0$ and $Z = X$!

Using these tools, we can deduce

Thm. $\varphi: X \to Y$ finite morphism $\Rightarrow$

$$\dim X \leq \dim Y.$$

If φ is also dominant, then

$$\dim X = \dim Y.$$

Proof. Replacing Y by $\overline{\varphi(X)}$, we reduce to proving the second statement.

Take $W_0 \subsetneq \dots \subsetneq W_r$ chain of closed irred in X.

Then $\varphi(W_0) \subset \dots \subset \varphi(W_r)$ is a chain in Y: irred - ok, closed - because φ finite.

Last Lemma $\Rightarrow$ inclusions are strict $\Rightarrow$

$$\dim X \leq \dim Y.$$

Conversely, a chain $Z_0 \subsetneq \dots \subsetneq Z_r$ in Y by Going Up gives a chain

$$W_0 \subsetneq \dots \subsetneq W_r \text{ in } X \text{ s.t. } \varphi(W_i) = Z_i.$$

build it from top: $W_r \rightsquigarrow W_{r-1} \rightsquigarrow \dots$

Hence $\dim X \geq \dim Y$.

(Define)
Recall: X irreducible variety $\rightsquigarrow$ its function field is

$$k(X) := \{(U, f) \mid U \overset{\neq\emptyset}{\subseteq} X \text{ open}, f \in \mathcal{O}_X(U)\} /_{\sim}$$

equal in a smaller nbhd

We call $f \in \mathcal{O}_X(U)$ rational functions on X.

Rem. 1) $U \subseteq X$ dense open $\Rightarrow k(U) \cong k(X)$.

2) We don't define $k(X)$ as $\operatorname{Frac} \mathcal{O}_X(X)$ because for projective varieties we'd get just k.

Ex. $k(\mathbb{P}^n) \cong k(\mathbb{A}^n) \cong k(x_1, .., x_n)$.

because $\mathbb{A}^n \subset \mathbb{P}^n$ is a dense open

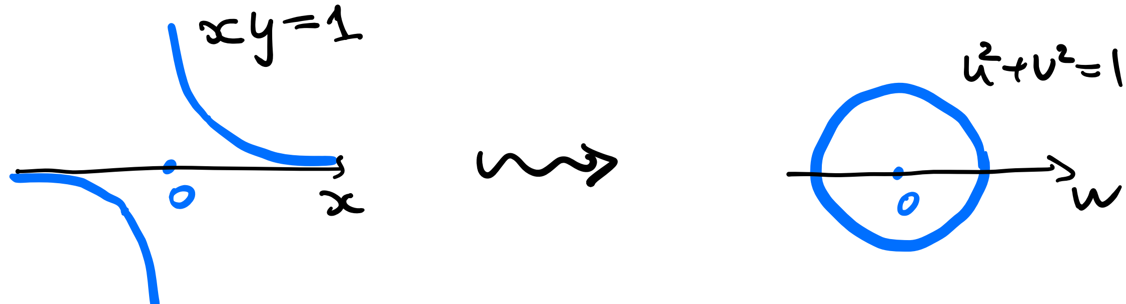
§Noether normalization

Recall **Noether normalization** lemma:

Thm. A fin. gen. k-algebra $\Rightarrow$
A is a finite extension of $k[x_1, \dots, x_d]$.
Moreover, for infinite k, $A \simeq k[z_1, \dots, z_n]/I$ $\rightsquigarrow$
x_i can be chosen as linear combinations of z_j.

Ex. $A = \mathbb{C}[x,y]/(xy-1) \simeq \mathbb{C}[x, x^{-1}]$
not finite over $k[x]$

Set $x := u + iv$
$y := u - iv$ $\rightsquigarrow$ $A = \mathbb{C}[u,v]/u^2+v^2-1$
finite over $k[u]$
because $v^2 + (u^2-1)$ is monic in v.



Cor. (Geometric interpretation, k infinite)
$X \subseteq \mathbb{A}^n$ affine $\Rightarrow \exists$ projection $\mathbb{A}^n \to \mathbb{A}^d$
linear map
which induces
a finite dominant morphism $X \xrightarrow[\varphi]{} \mathbb{A}^d$.

Morally: it's similar to L/k decomposing as $k \subset F \subset L$
pure transcendental, algebraic

We now can deduce the following important result.

Thm X irreducible variety $\Rightarrow$

$$\dim X = \operatorname{tr\,deg}_k k(X)$$

Proof. 1) X affine

Take $\varphi: X \to \mathbb{A}^d$ finite dominant $\Rightarrow$
$\dim X = \dim \mathbb{A}^d = d$ (as we proved before).
Moreover, $k[X] \supseteq k[x_1,..,x_d]$ is finite $\Rightarrow \operatorname{tr\,deg}_k k(X) = d$.

2) X general $\Rightarrow X = \bigcup_{i=1}^{n} U_i$ open affine cover.
As we saw, $\dim X = \sup \dim U_i$, and

$$\dim U_i = \operatorname{tr\,deg}_k k(U_i) = \operatorname{tr\,deg}_k k(X).$$

Cor. X irreducible variety, $U \subseteq X$ dense open $\Rightarrow$
$\dim U = \dim X$.

Proof. follows from Thm because
$k(U) = k(X)$.

§Codimension

def. X top space $Y \subseteq X$ closed, irreducible.
The codimension of Y in X is

$$\operatorname{codim}_X Y := \sup\{n \mid Y = X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_n \subseteq X\}$$
X_i closed irred

Rem. For any subset Y one can take $\min\limits_{Y_i \subset Y \text{ irred comps}} \operatorname{codim} Y_i$, but this notion behaves well only for irreducible sets.

Prop. 1) $\dim Y + \operatorname{codim}_X Y \leq \dim X$

2) X affine $\Rightarrow \operatorname{codim} Y = \operatorname{ht} I(Y) = \dim k[X]_p$
↑height of $p = I(Y)$

3) X irreducible variety $\Rightarrow$

$$\dim Y + \operatorname{codim}_X Y = \dim X.$$

Non-Ex: 

$X = \{p \cup l\}$

$\dim p = 0$, $\operatorname{codim}_X p = 0$, $\dim X = 1$.

Proof: 3) we can assume X affine
Then $k[X]$ is a f.g. k-algebra and an integral domain $\Rightarrow k[X]$ is a catenary ring, i.e. all maximal chains of prime ideals $\{p \subsetneq \dots \subset q\}$ have the same length for fixed p, q.

Prop. X, Y alg varieties $\Rightarrow$
$$\dim X \times Y = \dim X + \dim Y.$$

Proof sketch: • reduce to affines (exercise).
• X, Y affine, $\dim X = n$, $\dim Y = m \Rightarrow$
$\exists$ finite dominant maps (Noether's normalization)
$\varphi : X \to \mathbb{A}^n$, $\psi : Y \to \mathbb{A}^m$
$\rightsquigarrow \varphi \times \psi : X \times Y \to \mathbb{A}^n \times \mathbb{A}^m$, which is dominant.

Why $\varphi \times \psi$ is finite: it is the composition
$X \times Y \xrightarrow{\varphi \times id} \mathbb{A}^n \times Y \xrightarrow{id \times \psi} \mathbb{A}^n \times \mathbb{A}^m$,
and $\varphi \times id$ (also $id \times \psi$) is finite because
$k[X]$ finite $k[\mathbb{A}^n]$-module $\Rightarrow$
$k[X] \otimes_k k[Y]$ finite $k[\mathbb{A}^n] \otimes_k k[Y]$-module.

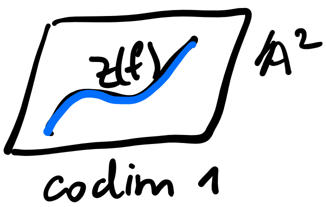
Hence $\dim X \times Y = \dim \mathbb{A}^n \times \mathbb{A}^m = n + m$.

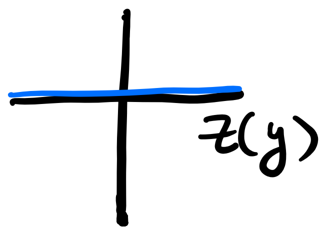
Thm (Krull's principal ideal thm)
A noetherian ring, $f \neq 0$, $f \notin A^\times$ $\Rightarrow$
$\forall$ minimal prime ideal $\mathfrak{p} \ni f$: $\operatorname{ht} \mathfrak{p} \leq 1$.
If f not a zero divisor $\Rightarrow \operatorname{ht} \mathfrak{p} = 1$.

Cor. X variety, $f \neq 0 \in \mathcal{O}_X(X)$. If $Z(f) \neq \emptyset$,
then $\operatorname{codim}_X Z \leq 1$ $\forall$ irred component Z of $Z(f)$

„an extra equation drops dimension at most by 1"

Ex.





has codim 0 in $Z(x \cdot y)$

NB: Not every codim 1 subvariety is cut out by one equation!

Let $X = Z(xy - zt) \subset \mathbb{A}^4$, irreducible, $\dim X = 3$.

Consider the plane $Z(x, z) \subset X$:
it has dim 2 $\Rightarrow$ codim 1 in X,
but it is not cut out by one equation.

Thm. X alg variety, $f_1, .., f_r \in \mathcal{O}_X(X)$.
Then every component Z of $Z(f_1, .., f_r)$ has codim $\leq r$.

Proof: • X irreducible. Want: $\dim Z \geq n-r$; $n = \dim X$.

Induction on r: $r=1$ - previous case

$r>1$: let W be a component of $Z(f_1, .., f_{r-1})$ that contains Z.

By induction, $\dim W \geq n-(r-1)$.

Z is a component of $W \cap Z(f_r)$

$f_r \equiv 0$ on $W \Rightarrow$ $Z = W$ and $\dim Z \geq n-r+1 \geq n-r$	$f_r \not\equiv 0$ on $W \Rightarrow$ $\dim Z = \dim W - 1$ by Krull's thm $\Rightarrow \dim Z \geq (n-r+1)-1 = n-r$.

• $X = \bigcup X_i$ irreducible comps

Fix Z $\rightsquigarrow$ choose i s.t. $Z \subseteq X_i$ and $\operatorname{codim}_{X_i} Z = \operatorname{codim}_X Z$ (i.e. codim in X_i is maximal).

Then Z is an irr comp of $Z(f_1, .., f_r) \cap X_i \Rightarrow$
$\operatorname{codim}_X Z = \operatorname{codim}_{X_i} Z \leq r$.

§ Systems of parameters

Morally: „local coordinates"

def. A local Noetherian ring of dim n, $\mathfrak{m} \subset A$.
A sequence $(f_1, \dots, f_n)$ in $\mathfrak{m}$ is a **system of parameters** if $(f_1, \dots, f_n)$ is $\mathfrak{m}$-primary, i.e. $\sqrt{(f_1, \dots, f_n)} = \mathfrak{m}$.

In other words, $A/(f_1, \dots, f_n)$ is Artinian ring.

Geometric interpretation

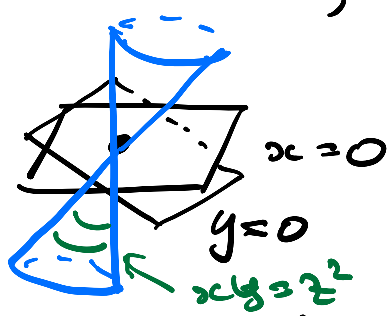
def. X affine var. of dim n, $x \in X$.
$(f_1, \dots, f_n) \in \mathcal{O}_{X,x} =: A$ is a **system of parameters** at x if all f_i's vanish at x and x is isolated (i.e., an irreducible comp.) in $Z(f_1, \dots, f_n)$.

By Nullstellensatz, it's the same def.

Ex. $A = k[x,y,z]/(xy - z^2)$, choose the point $(0,0,0)$

- (x,y) form a system of parameters at 0, because $\sqrt{(x,y)} = (x,y,z)$

- (x,z) does not –II–, because $A/(x,z) \simeq k[y]$, so $\mathfrak{p} = (x,z) \subset (x,y,z)$ is strict.



parabolas for fixed y

Prop. (they exist!)
X affine irreducible of dim n, $x \in X$.
Then $\exists f_1, .., f_n \in k[X]$ s.t. x is an isolated point in $Z(f_1, .., f_n) \subseteq X$.

Proof: skip
uses: Krull's thm and prime avoidance lemma (comm alg).

§ Applications

Fibers of morphisms

Prop. $\varphi: X \to Y$ dominant morphism of irreducible varieties, $y \in \varphi(X)$.

Then $\forall$ component Z of $\varphi^{-1}(y)$,

$$\dim Z \geq \dim X - \dim Y.$$

can be strict

Proof. Can assume X, Y affine, $r = \dim Y$.
Choose $(f_1, .., f_r)$ system of parameters at y.
By replacing Y with an affine open, we can assume $Z(f_1, .., f_r) = y$.

Hence $\varphi^{-1}(y) = Z(\varphi^* f_1, ..., \varphi^* f_r)$.

By generalized Krull's thm, every component of $\varphi^{-1}(y)$ has codim $\leq r$.

Hence $\dim X - \dim Z \leq r = \dim Y$.

A stronger statement is true, but harder to prove.

Thm. Under the same assumptions,
$\exists$ non-empty open $U \subseteq Y$ s.t. $\forall y \in U$
$\forall$ non-empty comp Z of $\varphi^{-1}(y)$: $\dim Z = \dim X - \dim Y$.

Intersections

Prop. (Affine case)
$X, Y \subseteq \mathbb{A}^n$ affine irreducible varieties.
Then $\forall$ non-empty component $Z \subseteq X \cap Y$

$$\operatorname{codim}_{\mathbb{A}^n} Z \leq \operatorname{codim}_{\mathbb{A}^n} X + \operatorname{codim}_{\mathbb{A}^n} Y$$

Proof. Consider the diagonal
$\{(x,x) \mid x \in X\} =: \Delta_X \hookrightarrow X \times X$. „reduction to the diagonal"

For $X, Y \subseteq \mathbb{A}^n$

$$X \cap Y = (X \times Y) \cap \Delta_{\mathbb{A}^n} \subset \underset{\{x_i\}}{\mathbb{A}^n} \times \underset{\{y_i\}}{\mathbb{A}^n}.$$

Hurrah! $\Delta_{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{A}^n$ is cut out by n equations $\{x_i - y_i\}_{i=1}^n$, and has codim n.

Hence $X \times Y \cap \Delta_{\mathbb{A}^n} \subset X \times Y$ is cut out by restrictions of these equations to $X \times Y$.

By Krull's thm, $\forall$ component $Z \subset X \cap Y$ satisfies

$$\dim Z \geqslant \dim(X \times Y) - n$$
$$\qquad\qquad \| $$
$$\dim X + \dim Y - n$$

$$\Rightarrow \operatorname{codim}_{\mathbb{A}^n} Z \leq \operatorname{codim}_{\mathbb{A}^n} X + \operatorname{codim}_{\mathbb{A}^n} Y.$$

Ex. $X, Y \subseteq \mathbb{A}^n$ is important, otherwise:

$Q := Z(xw - yz) \subset \mathbb{A}^4$ 3-dimensional

$Z_1 = (x, y)$ and $Z_2 = (z, w)$ intersect at $p = 0$.

We have: $\operatorname{codim}_Q p = 3$;

$\operatorname{codim}_Q Z_1 = \operatorname{codim}_Q Z_2 = 1$;

and $3 \not\leq 1 + 1$!

Prop. (projective case)

$X, Y \subseteq \mathbb{P}^n$ irreducible projective varieties such that $\operatorname{codim}_{\mathbb{P}^n} X + \operatorname{codim}_{\mathbb{P}^n} Y \leq n$.

Then:
1) $X \cap Y \neq \emptyset$
2) $\forall$ comp $Z \subseteq X \cap Y$
$\operatorname{codim}_{\mathbb{P}^n} Z \leq \operatorname{codim}_{\mathbb{P}^n} X + \operatorname{codim}_{\mathbb{P}^n} Y$.

Proof: Consider affine cones $C(X), C(Y) \subseteq \mathbb{A}^{n+1}$.

We have $\dim C(X) = \dim X + 1 \quad \forall X \subseteq \mathbb{P}^n$,

hence $\operatorname{codim}_{\mathbb{A}^{n+1}} C(X) = \operatorname{codim}_{\mathbb{P}^n} X$.

We know that $0 \in C(X) \cap C(Y)$.

For any component $Z \subset C(X) \cap C(Y)$

by previous Prop we have

$$\operatorname{codim}_{\mathbb{A}^{n+1}} Z \leq \operatorname{codim}_{\mathbb{A}^{n+1}} C(X) + \operatorname{codim}_{\mathbb{A}^{n+1}} C(Y) \leq n,$$

by assumption

so $\dim Z \geq 1 \Rightarrow$

$C(X) \cap C(Y) - 0$ is non-empty

$$\Rightarrow X \cap Y \neq \emptyset.$$

Moreover, $C(X) \cap C(Y)$ is the cone over $X \cap Y$,

so the desired inequality of codim follows from the previous Prop applied to the cones.

Ex. Again, $\subseteq \mathbb{P}^n$ is important, otherwise:

take $\mathbb{P}^1 \times \mathbb{P}^1$ with coords $(x_0 : x_1); (y_0 : y_1)$.

Consider $L_1 = Z_+(x_0)$ and $L_2 = Z_+(x_1)$.

We have

$$\operatorname{codim}_{\mathbb{P}^1 \times \mathbb{P}^1} L_1 + \operatorname{codim}_{\mathbb{P}^1 \times \mathbb{P}^1} L_2 = 1 + 1 = \dim \mathbb{P}^1 \times \mathbb{P}^1,$$

but $L_1 \cap L_2 = \emptyset$.