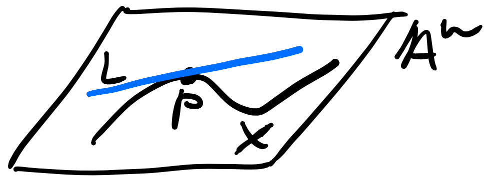
Chapter 6
Smoothness and singularities

§Tangent space

Intuitively:



We want to define the tangent space T_pX as the space of vectors that have a high order contact with X at p.

However, it's tricky because we don't want T_pX to depend on the embedding $X \hookrightarrow \mathbb{A}^n$, and we want it to be defined for non-affine varieties too.

Hypersurfaces in $\mathbb{A}^n$

Assume $X = Z(f) \subset \mathbb{A}^n$, $p = (a_1, \dots, a_n) \in X$

line passing through p: $L = \{(a_1 + tb_1, \dots, a_n + tb_n \mid t \in k\}$ ← parameter

def. L is tangent to X at p if the polynomial $g(t) := f(a_1 + tb_1, \dots, a_n + bt_n)$ has a multiple root at $t = 0$, i.e. $g'(t) = 0$.

↳ intersection multiplicity of X and L at p

Equivalently, $\sum b_i \cdot \frac{\partial f}{\partial x_i}(p) = 0$,

i.e. $(b_1, \dots, b_n)$ belongs to the tangent space

$$T_pX := \underbrace{\left\{ (v_1, \dots, v_n) \in k^n \;\middle|\; \sum_i v_i \cdot \frac{\partial f}{\partial x_i}(p) = 0 \right\}}_{\text{vector subspace of } k^n}$$

Affine varieties

$X \subseteq \mathbb{A}^n$ affine variety, $I(X) = (f_1, \dots, f_r)$

def. The tangent space of X at p is

$$T_pX := \left\{ (v_1, \dots, v_n) \in k^n \;\middle|\; \sum v_i \cdot \frac{\partial f}{\partial x_i}(p) = 0 \quad \forall f \in I(X) \right\}$$

enough for $f_1, \dots, f_r$

$= \ker J$, where

$$J := \left(\frac{\partial f_i}{\partial x_j}(p) \right)_{\substack{i = 1 \dots r \\ j = 1 \dots n}} \quad \text{– Jacobian matrix}$$

T_pX is a k-vector space,

$$\dim T_pX = n - \operatorname{rk} J.$$

def. The affine tangent space of X at p is

$$p + T_pX := \left\{ (v_1, \dots, v_n) \in \mathbb{A}^n \;\middle|\; \sum \frac{\partial f}{\partial x_i}(p) \cdot (v_i - a_i) = 0 \right\}$$

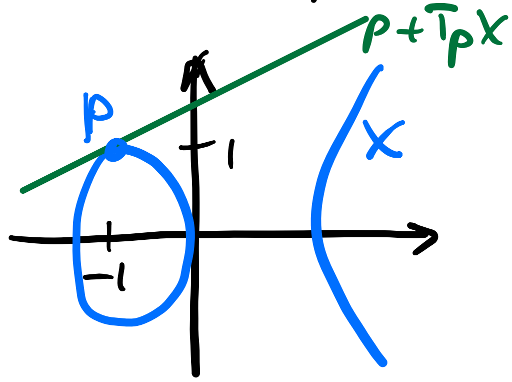
translation by p

Lemma. For $X \subseteq \mathbb{A}^n$ and $p=(0,\dots,0)$

$$p + T_pX = Z(\underbrace{f^{(1)}}_{\text{linear part of } f} \mid f \in I(X))$$

Ex. $X = Z(f) \subset \mathbb{A}^2$, $f = y^2 - x^3 + 2x$, $p=(-1,1)$.

$$\frac{\partial f}{\partial x}(p) = -3x^2+2\Big|_{x=-1} = -1;$$



$$\frac{\partial y}{\partial x}(p) = 2y\Big|_{y=1} = 2;$$

$$T_pX = \{(x,y)\in k^2 \mid \underbrace{(-1\ \ 2)}_{= J}\begin{pmatrix} x\\ y\end{pmatrix} = 0\} = \{(2t,t) \mid t \in k\}$$

Intrinsic description of T_pX

Want: get rid of the data of $\hookrightarrow \mathbb{A}^n$

wlog

Assume $X \subseteq \mathbb{A}^n$, $p=(0,\dots,0)$, $I := I(X)$.

Let $M := (x_1,\dots,x_n) \subset k[x_1,\dots,x_n]$.

For $f \in k[x_1,\dots,x_n]$ denote

$$f^{(1)} := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(0)\cdot x_i$$ – <u>linearization</u> of f at p

$f^{(1)}$ gives a linear function $k^n \to k$.

We get a k-linear map

$$\begin{aligned} d: \quad M &\longrightarrow (k^n)^\vee \\ \rotatebox{90}{$\in$} \quad & \qquad \rotatebox{90}{$\in$} \\ f &\longmapsto f^{(1)} \end{aligned}$$

Observe:

- d is surjective, since $dx_i = x_i$
- $f \in \ker d \Rightarrow$ all terms in f have $\deg \geq 2$ $\Rightarrow f \in M^2$

We get the following

Lemma. d induces an isom

$$M/_{M^2} \xrightarrow{\sim} (k^n)^\vee$$

We had: $T_p X \hookrightarrow k^n$.

Dualize: $(k^n)^\vee \twoheadrightarrow (T_pX)^\vee$

Obtain:

$$\begin{tikzcd} M/_{M^2} \arrow[r, "\simeq"] \arrow[rr, bend right, "\theta"'] & (k^n)^\vee \arrow[r, two heads] & (T_pX)^\vee \end{tikzcd}$$

θ - surjective map

Lemma. $\ker \theta = M^2 + I$.

Proof: $f \in \ker \theta \iff f^{(1)} = 0$ on T_pX

$\iff f^{(1)} = g^{(1)}$ for some $g \in I$

Lemma $\iff f - g \in \ker d = M^2 \iff f \in M^2 + I$.

We get altogether:

$$(T_pX)^\vee \simeq M/_{M^2+I} \simeq \frac{M/I}{M^2+I/I} \simeq M/I /_{M^2/I},$$

and the latter is $m/_{m^2}$ for $m \subset \mathcal{O}_{X,p}$ the max ideal in the local ring $\mathcal{O}_{X,p}$.

By taking duals, we obtain

Prop. $\forall$ affine variety $X \subseteq \mathbb{A}^n$, there is a natural isomorphism

k-vector space

$$T_pX \simeq \left(m_p/_{m_p^2}\right)^\vee = \mathrm{Hom}_k\left(m_p/_{m_p^2}, k\right)$$

intrínsic invariant of X!

Tangent spaces in general

def. X alg variety, $p \in X$.
The (Zariski) tangent space of X at p is

$$T_pX := \mathrm{Hom}_k\left(m_p/_{m_p^2}, k\right), \quad m_p \subset \mathcal{O}_{X,p}.$$

Functoriality $\varphi : X \to Y$ morphism of varieties

$$\leadsto \varphi^* : \mathcal{O}_{Y,\varphi(p)} \to \mathcal{O}_{X,p}$$

φ^* is a local hom. $\leadsto$

$$m_{\varphi(p)}/m^2_{\varphi(p)} \longrightarrow m_p/m_p^2$$

$(-)^\vee$ $\leadsto$

$$d\varphi_p : T_pX \to T_{\varphi(p)}Y$$

The map df_p is called the **differential of φ at p.**

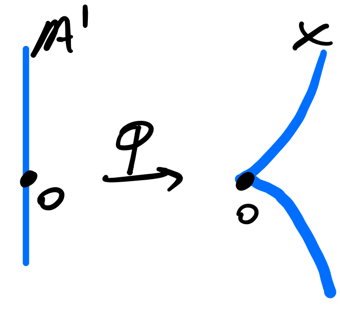
Lemma. If $\varphi : X \to Y$ is a polynomial map induced by $\mathbb{A}^n \to \mathbb{A}^m$ $x \mapsto (f_1(x), \dots, f_m(x))$, then $d\varphi_p$ is the multiplication by

$$\mathcal{J}(\varphi) := \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{i=1\dots m \\ j=1\dots n}}$$

Jacobian matrix

Ex. cuspidal cubic

$$\varphi : \mathbb{A}^1 \longrightarrow C = Z(y^2 - x^3) \subset \mathbb{A}^2$$
$$t \longmapsto (t^2, t^3)$$



We have $\varphi^* : k[x,y]/_{y^2 - x^3} \to k[t]$

$$x \longmapsto t^2$$
$$y \longmapsto t^3$$

Let's compute differentials.

- $p = 0$, so $\varphi(p) = (0,0)$.

Then $\mathcal{O}_{C,\varphi(p)} \to \mathcal{O}_{\mathbb{A}^1,p}$ is just the localization of φ^* at (x,y) resp. (t).

It restricts to:

$$(x,y)/(x,y)^2 + I(X) \simeq (x,y)/(x^2, xy, y^2, y^2 - x^3) \longrightarrow (t)/(t^2)$$

which sends $x \mapsto t^2$ and $y \mapsto t^3$.

Hence $d\varphi_p : T_p\mathbb{A}^1 \to T_pC$ is the zero map.

- $q \neq 0 \in \mathbb{A}^1 \Rightarrow d\varphi_q$ is an isom (check by hand).

„Cusp is a singularity!"

§ Singularities

We want to define singularities in terms of the intrinsic def. of tangent space.

Lemma. (A, m) Noetherian local ring, $k = A/m \Rightarrow$

$$\dim_k m/m^2 \geq \dim A.$$

Proof: pick a basis $v_1, \dots, v_r$ of k-vector space m/m^2; by Nakayama's lemma we can lift it to a set of generators $(w_1, \dots, w_r) = m$.

By Krull's principal ideal theorem, $\operatorname{ht} m \leq r$, and $\dim A = \operatorname{ht} m$.

def. (A, m) Noetherian local ring is **regular** if $\dim_k m/m^2 = \dim A$.

Cor. X alg variety, $p \in X \Rightarrow$

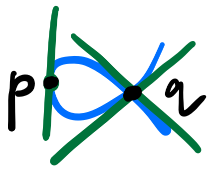
$$\dim T_p X \geq \dim \mathcal{O}_{X,p}$$

def. We call a point $p \in X$ **non-singular** or **regular** if $\dim T_p X = \dim \mathcal{O}_{X,p}$

i.e. if the local ring of X at p is regular.

Otherwise, we call p singular.

$\mathrm{Sing}(X) := \{p \in X \mid p \text{ singular}\}$ is the singular locus of X.



p regular

q singular – more tangent directions!

def. We say that X is non-singular or regular if

$$\mathrm{Sing}(X) = \emptyset.$$

Rem. for future Singularities may appear after passing to a bigger base field, in general.

The reason is that, say,

$k = \mathbb{F}_p(t) \implies (x^p - t) \subset k[x]$ is radical,

but $(x^p - t) = (x - t^{\frac{1}{p}})^p \subset \bar{k}[x]$ is not radical, and that's how singularities can arise (see later in scheme theory)

def. We call $\dim \mathcal{O}_{X,p}$ the dimension of X at p, since

$$\dim \mathcal{O}_{X,p} = \max_{\substack{X_i \ni p \\ \text{irred comp}}} \dim X_i.$$

X irreducible $\implies \dim_p X = \dim X \quad \forall p.$

$\dim_p X = 1$

$\dim_q X = 2$

Fact. (A, m) regular local ring. Then A is an integral domain.

Cor. $p \in X$ regular $\Rightarrow$ $\exists!$ irreducible comp. $X_i \subseteq X$ passing through p.

Jacobian criterion. $X \subseteq \mathbb{A}^n$ affine, $p \in X$, $\dim_p X = d$. Then X is non-singular at p iff the Jacobian matrix J has rank $n-d$ at p.

$(f_1, .., f_r) = I(X)$ "$\left(\frac{\partial f_i}{\partial x_j}\right)$" maximal rank

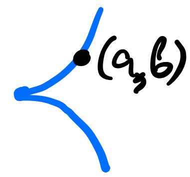
Rem. Same condition for implicit function thm in Analysis!

Prop. $\mathrm{Sing}(X)$ is a closed subset of X.

Proof: Enough for affine X ($X - \mathrm{Sing}(X)$ is open).

Then $p \in \mathrm{Sing}(X)$ means that the Jacobian matrix J at p does not have full rank, which is a closed condition. ∎

Moreover, $\forall r \geq 0$ $\{p \in X \mid \dim T_p X \geq r\}$ is a closed set, determined by vanishing of $(n-r+1) \times (n-r+1)$ minors of J.



Ex. $X = Z(\overbrace{y^2 - x^3}^{f}) \subset \mathbb{A}^2$

At (a,b): $J = \left(\frac{\partial f}{\partial x}(p) \quad \frac{\partial f}{\partial y}(p) \right) = (3a^2 \quad -2b)$

$$p + T_pX = Z\big(3a^2(x-a) - 2b(y-b)\big) \subset \mathbb{A}^2$$

X is non-singular at $p \iff T_pX$ has $\dim = 1$
$\iff (a,b) \neq (0,0)$

Computing m_p/m_p^2

$M = (x-a, y-b)$; $I(X) = (y^2 - x^3) \implies$

$$m_p/m_p^2 = \frac{(x-a, y-b)}{(x-a)^2, (x-a)(y-b), (y-b)^2, y^2 - x^3} \underset{\text{change of coords}}{\simeq}$$

$$\frac{(x, y)}{(x^2, xy, y^2, (x+a)^3 - (y+b)^2)} = \frac{(x,y)}{(x^2, xy, y^2, \underbrace{3a^2x + a^3 - 2by - b^2}_{g})}$$

$p = (0,0) \implies m/m^2 \simeq kx \oplus ky \implies \dim_k m/m^2 = 2$

$p \neq (0,0) \implies g(x,y)$ is a linear equation,
so $\dim_k m/m^2 = 1$

char $k \neq 2,3$: $m/m^2 = (\bar{x}) = (\bar{y})$
char $k = 2$: $m/m^2 = (\bar{y})$; char $k = 3$: $m/m^2 = (\bar{x})$.

Alternative description of T_pX

def. Let A be a k-algebra, M an A-module. A k-derivation of A into M is a k-linear map
$$D : A \to M, \quad \text{satisfying}$$
Leibniz rule: $\quad D(fg) = fD(g) + gD(f).$

In particular, $\forall\, c \in k \quad D(c) = 0$ because
$$D(1) = D(1\cdot 1) = D(1)\cdot 1 + 1\cdot D(1) = 2\cdot D(1).$$

The space of k-derivations of A into M

$\mathrm{Der}_k(A, M)$ is a k-vector space.

Our case: X alg variety, $p \in X \rightsquigarrow$
$A = \mathcal{O}_{X,p}$, $M = k$ is an A-module via $\mathcal{O}_{X,p} \to k$
$$f \mapsto f(p)$$

def. The differential map is the map
$$d_p : \mathcal{O}_{X,p} \to m_p/m_p^2$$
$$f \mapsto d_pf := f - f(p) \mod m_p^2$$

Lemma. d_p is k-linear and satisfies Leibniz rule.

Ex. $X = \mathbb{A}^n$, $p = (0,\dots,0) \Rightarrow \{dx_i\}_{i=1}^n$ form a basis of the cotangent space m/m^2, and
$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(0)\cdot dx_i \qquad \forall f \in k[x_1,\dots,x_n].$$

Thm. There is a canonical isom of k-vector sp.:

$$\mathrm{Der}_k(\mathcal{O}_{X,p}, k) \xrightarrow[\sim]{\theta} T_pX$$

Proof. • define θ

we can decompose $\mathcal{O}_{X,p}$ as k-vector space:

$$\mathcal{O}_{X,p} \simeq k \oplus m_p$$
$$f \longmapsto (f(p), f-f(p))$$

Pick a derivation $D: \mathcal{O}_{X,p} \to k$

zero on constants $\Big\downarrow$

$$D: m_p \to k$$

Leibniz rule $\Big\downarrow$

$$\underset{\substack{\cap \\ m^2}}{D(fg)} = \underset{\substack{\| \\ 0}}{f(p)} \cdot D(g) + \underset{\substack{\| \\ 0}}{g(p)} D(f) \qquad m_p/m_p^2 \to k$$

$$\therefore\; \theta(D) \in T_pX = \mathrm{Hom}_k(m_p/m_p^2, k)$$

• inverse map

Given any k-linear $v: m_p/m_p^2 \to k$,

we define $D = v \circ d_p : \mathcal{O}_{X,p} \xrightarrow{d_p} m_p/m_p^2 \xrightarrow{v} k$

derivation by the lemma. $f \longmapsto \overline{f-f(p)} \mapsto v(\overline{f-f(p)})$.

Rem. Morally, Θ^{-1} sends a tangent vector v to $\frac{d}{dv}$ – the „directional derivative".

More concretely, if we replace $\mathcal{O}_{X,p}$ with $C^\infty_p(\mathbb{R}^n)$, $p \in \mathbb{R}^n$ (stalk of smooth functions at p),

then $\mathrm{Der}_{\mathbb{R}}(C^\infty_p(\mathbb{R}^n), \mathbb{R}) \simeq \mathbb{R}^n$

$$\partial/\partial v \longleftarrow v$$

where $$\frac{\partial}{\partial v}(f) := \frac{d}{dt}\big(t \mapsto f(p+tv)\big)(0)$$

„directional derivative along v"

analytic derivative

§ Tangent spaces for projective varieties

First, let $X = Z_+(F) \subset \mathbb{P}^n$ be a hypersurface and $p = (1 : \omega_1 : \ldots : \omega_n) \in D_+(x_0) \cap X$.

Then in $D_+(x_0) \simeq \mathbb{A}^n$ we have

$$p + T_p X = \{(z_1, \ldots, z_n) \in \mathbb{A}^n \mid \sum_{i=1}^n \frac{\partial f}{\partial z_i}(p)(z_i - \omega_i) = 0\}$$

where $f(z_1, \ldots, z_n) = F(1, x_1, \ldots, x_n)$.

We define the projective tangent space as the closure of $p + T_p X$ in $\mathbb{P}^n$:

$$\mathbb{T}_p X = \{(x_0 : \ldots : x_n) \in \mathbb{P}^n \mid \sum_{i=1}^n \frac{\partial F}{\partial x_i}(1, \omega_1, \ldots, \omega_n) \cdot (x_i - \omega_i x_0) = 0\}$$

Euler's formula: F homogeneous of deg $d \Rightarrow$

$$d \cdot F = \sum_{j=0}^n x_j \cdot \frac{\partial F}{\partial x_j} \qquad \text{(check for monomials)}$$

We have: $F(1, \omega_1, \ldots, \omega_n) = 0$.

Hence $\sum_{i=1}^n \frac{\partial F}{\partial x_i}(1, \omega_1, \ldots, \omega_n)(-\omega_i x_0) = \frac{\partial F}{\partial x_0}(1, \omega_1, \ldots, \omega_n) x_0$.

$$\Rightarrow \mathbb{T}_p X = \{(x_0 : \ldots : x_n) \mid \sum_{i=0}^n \frac{\partial F}{\partial x_i}(p) \cdot x_i = 0\}$$

def. $X \subseteq \mathbb{P}^n$ projective variety, $p \in X$.
The projective tangent space of X at $p \in X$ is

$$\mathbb{T}_p X := \left\{ (x_0 : \ldots : x_n) \in \mathbb{P}^n \;\middle|\; \sum_{i=0}^{n} \frac{\partial F}{\partial x_i}(p) \cdot x_i = 0 \quad \forall F \in I(X) \right\}$$

Projective Jacobian criterion (follows from affine)

Let $X = Z_+(\underbrace{F_1, \ldots, F_r}_{\text{radical ideal}}) \subseteq \mathbb{P}^n$, $p \in X$,

$$J = \left(\frac{\partial F_i}{\partial x_j}(p) \right)$$ pick a representable of p

Then: • $\operatorname{rk} J$ does not depend on the representative of p

• X is non-singular at p
iff $\operatorname{rk} J = n - \dim X$

Ex. Fermat hypersurface

$X = Z_+(F) \subset \mathbb{P}^n$, $F = x_0^p + \ldots + x_n^p$

$J = (p x_0^{p-1} \; p x_1^{p-1} \; \ldots \; p x_n^{p-1})$

• char $k \neq p \Rightarrow \operatorname{rk} J = 1$ everywhere $\Rightarrow$ X is regular

• char $k = p \Rightarrow$ Jacobian criterion fails because $F = (x_0 + \ldots + x_n)^p$ – not radical. Instead:
$X = Z_+(x_0 + \ldots + x_n) \cong \mathbb{P}^{n-1}$ also regular!

§ Singular locus

Thm. X alg variety $\Rightarrow X - \mathrm{Sing}(X)$ is dense in X.

Enough to prove for every irreducible component, so we can assume X irreducible, and then we just need to show that $X - \mathrm{Sing}(X) \neq \emptyset$ (we know it's open).

We first consider the case of a hypersurface in an affine space.

Prop. $Y \overset{\neq\emptyset}{=} Z(f) \subsetneq \mathbb{A}^n$ irreducible hypersurface. Then $\mathrm{sing}(Y) \subset Y$ is a proper closed subset.

Proof: $p \in \mathrm{Sing}\, Y \iff \forall i \ \frac{\partial f}{\partial x_i} = 0$.

Hence $\mathrm{Sing}(Y) = Y \iff \frac{\partial f}{\partial x_i} \in I(Y) = (f) \ \forall i$.

But (f) was prime and $\deg \frac{\partial f}{\partial x_i} < \deg f$.

Hence $\frac{\partial f}{\partial x_i} \equiv 0 \ \forall i$.

- $\mathrm{char}\, k = 0 \Rightarrow f$ is constant $\Rightarrow Y = \emptyset$.
- $\mathrm{char}\, k = p \Rightarrow f$ is a polynomial in $\{x_i^p\}_{i=1}^n$

$\Rightarrow$ $f = g^p$ for $g \in k[x_1, .., x_n] \Rightarrow f$ not irreducible.

uses k perfect

To prove the general thm, we need the notion of birationality.

def. Two varieties X, Y are birational if $\exists$ open dense $V \subseteq X$, $W \subseteq Y$ s.t. $V \simeq W$.

Ex: $\mathbb{A}^n \underset{Bir}{\sim} \mathbb{P}^n$

Lemma. X, Y birational varieties. If X admits a regular point, then so does Y.

Proof: Let $\varphi : V \xrightarrow{\sim} W$; $V \subseteq X$, $W \subseteq Y$ dense open.

Then $U := X - \mathrm{Sing}(X) \subseteq X$ non-empty open $\Rightarrow$

$\underset{\neq \emptyset}{U \cap V} \xrightarrow{\sim} \varphi(U \cap V) \subseteq W$ – non-empty open.

$\mathrm{Sing}(U \cap V) = \emptyset \Rightarrow \mathrm{Sing}(\varphi(U \cap V)) = \emptyset$.

The Thm then follows from the following claim, that we'll prove later.

Prop. Any irreducible variety of dim n is birational to an irreducible hypersurface in $\mathbb{A}^{n+1}$.

§ Normal varieties

<u>def.</u> An integral domain R is **integrally closed** if $\forall a \in \mathrm{Frac}(R)$ s.t. $\exists m \in \mathbb{N}$, $b_i \in R$

$$a^m + b_{m-1} a^{m-1} + \dots + b_1 a + b_0 = 0 \quad \Rightarrow a \in R.$$

<u>def.</u> An alg variety X is **normal** if $\forall x \in X$ the local ring $\mathcal{O}_{X,x}$ is an integral domain which is integrally closed.

In particular normal connected $\Rightarrow$ irreducible.

Lemma. X affine, then X normal iff $k[X]$ is integrally closed:

$\Leftarrow$ localizations of int. closed are int. closed

$\Rightarrow$ $k[X] = \bigcap_m k[X]_m$ and intersections are int. closed.

Fact: Regular varieties are normal (regular local ring $\Rightarrow$ ufd $\Rightarrow$ int closed)

Morally: normal varieties can have singularities but they are not too bad, and later we will learn how to approximate any variety by a normal one (via „normalization").

Ex: $X = Z(xy - z^2) \subset \mathbb{A}^3$ has a singularity at 0, but it is normal.

Prop. X normal variety, $Z \subset X$ closed subvariety, $\operatorname{codim}_X Z \geq 2$. Then any $f \in \mathcal{O}_X(X-Z)$ extends to a regular function on X.

Ex. $k[\mathbb{A}^n - 0] \simeq k[\mathbb{A}^n]$ $n \geq 2$.

Proof. We can assume X connected ($\Rightarrow$ irred) and affine (regularity is a local condition).

„Algebraic Hartog's lemma":
A Noetherian integrally closed integral domain

$$\Rightarrow \qquad A = \bigcap_{\operatorname{ht} p = 1} A_p \subset \operatorname{Frac}(A).$$

Morally: „if you have a fraction whose denominator is in none of the ht $=1$ prime ideals, then the denominator is invertible"

Now $\operatorname{codim}_X Z \geq 2 \Rightarrow$ Inside $k(X)$ we have

$$(*) \qquad \mathcal{O}_X(X-Z) \subset k[X]_p \quad \forall \operatorname{ht} p = 1$$

because $\underbrace{Z(p)}_{\text{codim } 1} \not\subset \underbrace{Z}_{\text{codim} \geq 2}$.

Hence $f \in \mathcal{O}_X(X-Z) \Rightarrow f \in \bigcap_{\operatorname{ht} p = 1} k[X]_p = k[X]$.

$k[X]$ is int. closed by the Lemma

$(*)$: $X - Z = \bigcup_i D(f_i)$, $\exists f_i \notin p$

$\varphi \in \mathcal{O}_X(X-Z) \Rightarrow \varphi \in \mathcal{O}_X(D(f_i)) = k[X]_{f_i} \hookrightarrow k[X]_p$

Prop. X normal variety, $Z \subset X$ closed irred, $\operatorname{codim}_X Z = 1$
$\Rightarrow \exists$ affine open $W \subseteq X$ s.t. $W \cap Z \neq \emptyset$ and
$I(Z \cap W) = (f) \subset k[W]$ for some $f \in \mathcal{O}_X(W)$.

Proof: can assume X affine connected $\Rightarrow$ irred.
Then $I(Z) = \mathfrak{p}$ for $\mathfrak{p} \lhd k[X]$, $\operatorname{ht} \mathfrak{p} = 1$.

Consider $k[X]_{\mathfrak{p}}$:

- it's integrally closed as localization of $k[X]_m$ for any $m \leftrightarrow p \in Z$;
- $\dim k[X]_{\mathfrak{p}} = 1$.

Hence $k[X]_{\mathfrak{p}}$ is a dvr $\Rightarrow$ it's a PID.
We conclude that the maximal ideal

$$\mathfrak{p} \cdot k[X]_{\mathfrak{p}} = (f) \quad \text{for} \quad f = \frac{g}{h} \in k(X),\ h \notin \mathfrak{p}.$$

We have $\mathfrak{p} = (s_1, \dots, s_r) \Rightarrow$
$\forall i \quad s_i = f \cdot \frac{u_i}{t_i}$ for $t_i \notin \mathfrak{p}$.
We let $W := D(h \cdot t_1 \cdot \ldots \cdot t_r)$, then $f \in k[W]$
and $\frac{u_i}{t_i} \in k[W] \Rightarrow \mathfrak{p} \cdot k[W] = (f)$.

Using this, we can prove the following.

Thm. X normal variety $\Rightarrow$
$\mathrm{Sing}(X)$ has codimension ≥ 2.

Cor. A singular curve cannot be normal.

Rem. Prop + Thm is a criterion of normality!

Proof. Assume $W \subset \mathrm{Sing}(X)$ irreducible, $\mathrm{codim}_X W = 1$.
By previous Prop. we can shrink X so that X is affine irreducible and
$$I(W) = (f), \quad f \in k[X].$$

variety

We know: $\exists\, w \in W$ regular point in W.

Nakayama $\Rightarrow$ $m_w \subset \mathcal{O}_{W,w}$ is gen. by $g_1, \ldots, g_{d-1}$, $d = \dim X$.

Since $\mathcal{O}_{W,w} = \mathcal{O}_{X,w}/(f)$,
we can lift g_i to $\tilde{g}_i \in \mathcal{O}_{X,w}$ and
$$(f, \tilde{g}_1, \ldots, \tilde{g}_{d-1}) = m_x \subset \mathcal{O}_{X,w}$$
$\Rightarrow \mathcal{O}_{X,w}$ is regular, so $w \notin \mathrm{Sing}(X)$!

Hurrah!