# Chapter 7
# Birationality

NB: in this chapter all varieties are by default assumed to be irreducible and separated!

def. A variety $X$ is separated if $\Delta_X \subset X \times X$ is closed.

Lemma $X$ separated $\iff$ $\forall \varphi, \psi : Z \to X$ locus $\{\varphi = \psi\}$ is closed.

## § Rational maps

invented way before regular maps!

def. A rational map $X \dashrightarrow Y$ of varieties is a morphism $\varphi : U \to Y$ defined on $U \subseteq X$ open.

A birational map is $\phi : X \dashrightarrow Y$ s.t. $\exists\, \psi : Y \dashrightarrow X$ s.t. $\phi \circ \psi = id$ and $\psi \circ \phi = id$ where these maps are defined.

In other words, it's an isom. $\phi : U \xrightarrow{\sim} V$ for $U \subseteq X$, $V \subseteq Y$ open.

We identify rational maps that agree on some non-empty open subset.

**def.-Lemma.** Given $\varphi : X \dashrightarrow Y$ $\exists! U_\varphi \subseteq X$ that is the maximal subset of definition of $\varphi$.

**Proof:** • $\exists$: $\Sigma := \{V \subseteq X \text{ open} \mid \varphi \text{ extends to a morphism on } V\}$

$X$ Noetherian $\Rightarrow$ $\Sigma$ has a maximal element

• !: assume both $\varphi_1 : V_1 \to Y$, $\varphi_2 : V_2 \to Y$ extend $\varphi : U \to Y$.

Want: $\varphi_1 = \varphi_2$ on $V_1 \cap V_2$, then we can extend $\varphi$ to $V_1 \cup V_2$ by gluing, hence $V_1 = V_2$ as they are both maximal.

But we know that $\varphi_1 = \varphi_2$ on $U \cap V_1 \cap V_2$. $Y$ separated $\Rightarrow$ by def., the locus $\{\varphi_1 = \varphi_2\}$ is closed in $V_1 \cap V_2$ $\Rightarrow$ it must be whole $V_1 \cap V_2$.

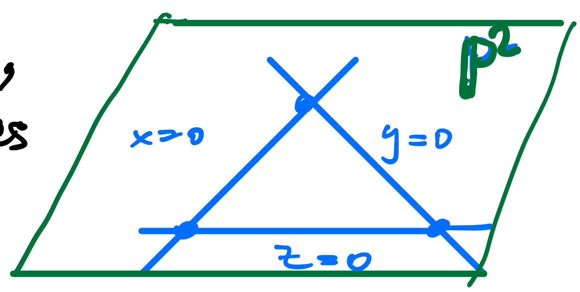
**Ex:** Cremona transformation (quadratic transform)

$$\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \quad (x:y:z) \longmapsto (xy : xz : yz),$$

$$U_\varphi = \mathbb{P}^2 - ((1:0:0) \cup (0:1:0) \cup (0:0:1)).$$

But on $D_+(xyz)$: $\varphi(x:y:z) = (z^{-1} : y^{-1} : x^{-1})$,

hence $\varphi \circ \varphi = \text{id}$ on $D_+(xyz)$, not on $U_\varphi$, since $\varphi$ collapses $Z_+(x)$ to $(0:0:1)$.



**Rem:** 1) $\varphi: X \dashrightarrow Y \rightsquigarrow \varphi(U_\varphi) \subseteq Y$ doesn't have to be open, so a priori we can't compose $\varphi: X \dashrightarrow Y$ and $\psi: Y \dashrightarrow Z$ because it may be that $\varphi(U_\varphi) \cap U_\psi = \emptyset$. But this is not an issue when $\varphi$ is dominant, i.e. $\varphi(U_\varphi) \subseteq Y$ is dense.

2) $\{\varphi: X \dashrightarrow \mathbb{A}^1\} = \{\varphi \in k(X)\}$, so rational maps to correspond to rational functions, hence the name.

In particular, $\varphi: X \dashrightarrow Y$ rational dominant induces $\varphi^*: k(Y) \to k(X)$ $f \mapsto f \circ \varphi$.

**def.** The category $\mathrm{Rat}_k$ consists of irreducible $k$-varieties and dominant rational maps between them.

**Thm.** There is an equivalence of categories:

$$\mathrm{Rat}_k^{op} \simeq \text{fin. gen. field extensions of } k$$

$$X \longmapsto k(X)$$

$$\varphi: X \dashrightarrow Y \longmapsto \varphi^*: k(Y) \to k(X)$$

**Cor.** $X \underset{bir}{\simeq} Y$ iff $k(X) \simeq k(Y)$ as $k$-algebras. ($k$-extensions)

**Rem.** It's important that $k(X) \simeq k(Y)$ as $k$-extensions:
$\overline{\mathbb{C}(x)}$ and $\mathbb{C}$ are isom as fields but not as $\mathbb{C}$-algebras.

**Ex.** $\mathbb{P}^2 \underset{bir}{\sim} \mathbb{P}^1 \times \mathbb{P}^1$ because $k(\mathbb{P}^2) \simeq k(\mathbb{P}^1 \times \mathbb{P}^1) \simeq k(x,y)$.
Both have $\mathbb{A}^2$ as a dense open subset

**Proof.** • $\forall F/k$ fin. gen. $\exists X: F = k(X)$
holds because $F = \mathrm{Frac}(R)$, $R$ reduced fin. gen. $k$-alg.
• given $k$-algebra hom $\alpha: k(Y) \to k(X)$
want: $\varphi: X \dashrightarrow Y$ s.t. $\alpha = \varphi^*$.
Choose any affine opens $U \subseteq X$ and $V \subseteq Y$.
Let $A = \mathcal{O}_X(U) \subset k(X)$, $B = \mathcal{O}_Y(V) \subset k(Y)$.

A priori, $\alpha(B) \not\subseteq A$, so we want to replace $A$ with some $A_a = D(a)$ so that $\alpha(B) \subseteq A_a$.
Choose $b_1, \dots, b_s$ generators of $B$ as a $k$-algebra, they give $\alpha(b_1), \dots, \alpha(b_s) \in k(X)$.
Write $\alpha(b_i) = \dfrac{a_i}{a} \in k(X)$, $a_i, a \in A$,
it's possible because $k(X) = \mathrm{Frac}(A)$.

Hence $\alpha: B \to A_a \subseteq k(X)$, and we can choose $U := D(a) \subseteq X$.
Main theorem of affine varieties gives a unique morphism $\varphi: D(a) \to V$ s.t. $\varphi^* = \alpha$
(other $U, V \rightsquigarrow$ same rational map) $\varphi: X \dashrightarrow Y$ rational map

Now we can prove the promised Prop.

**Prop.** Any variety of dim $n$ is birational to a hypersurface in $\mathbb{A}^{n+1}$.

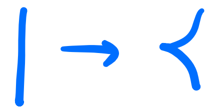
**Proof:** $n = \dim X = \operatorname{tr\,deg}_k k(X)$, *
hence $k(X)/k(x_1,..,x_n)$ is a finite separable extension.
By the Primitive element thm,
$\exists\, y \in k(X)$ algebraic over $k(x_1,..,x_n)$ s.t.
$$k(X) = k(y, x_1,..,x_n).$$
$y$ is algebraic $\Rightarrow$ $\exists$ minimal polynomial
$$y^m + a_{m-1} y^{m-1} + \ldots + a_0 = 0, \quad a_i \in k(x_1,..,x_n).$$
clear denominators $\Rightarrow$
$$f(y, x_1,..,x_n) = 0, \quad f \text{ irreducible}.$$
$Z(f) \subset \mathbb{A}^{n+1}$ is a hypersurface s.t.
$$k(Z(f)) \simeq k(X) \text{ as extensions of } k$$
$\Rightarrow$ by Corollary, $X \underset{\text{bir}}{\simeq} Z(f)$.

*: uses that every extension of a perfect (e.g. algebraically closed) field is separable (see Lang „Algebra" VIII, Cor. 4.4)

**def.** A normalization of a variety $X$ is a normal variety $\tilde{X}$ with a finite birational morphism (defined on whole $\tilde{X}$) $\pi : \tilde{X} \to X$ that is universal:

$\forall$ dominant $\varphi : Y \to X$ where $Y$ is normal, factors through $\pi$.

**Thm.** $\forall$ variety $X$ $\exists!$ normalization $\pi : \tilde{X} \to X$ (unique up to isom).



**Proof for affines** (in general: gluing... lengthy)

$A := k[X]$, $B :=$ integral closure of $A$ in $\mathrm{Frac}(A) = k(X)$.

**Fact** (CA): $B$ is a finite $A$-module.
In particular, $B$ is a f.g. $k$-algebra, so $B = k[\tilde{X}]$ and $\pi : \tilde{X} \to X$ is finite.

Since $\pi^* : k(X) \to k(\tilde{X})$ is an isom, $\pi$ is birational.

Universality:
$\varphi : Y \to X$ dominant $\Rightarrow k(X) \subseteq k(Y)$
$Y$ normal $\Rightarrow k[Y] \subset k(Y)$ integrally closed.

Hence $\forall \alpha \in k(X)$ integral over $k[X]$ $\alpha \in k[Y]$, so $k[\tilde{X}] \subseteq k[Y]$ giving the factorization $Y \to \tilde{X}$.

## § Birational automorphisms

Birational automorphisms give an important invariant:

$$X \mapsto \mathrm{Bir}(X) := \{\varphi : X \dashrightarrow X \text{ birational}\}$$

a group under composition!

Rem. 1) $\mathrm{Bir}\, X \not\cong \mathrm{Bir}\, Y \Rightarrow X \underset{\mathrm{bir}}{\not\sim} Y$

2) $\underset{\{\varphi : X \xrightarrow{\sim} X\}}{\mathrm{Aut}(X)} \leq \mathrm{Bir}(X)$

subgroup

$\mathrm{Bir}\, \mathbb{P}^n$ is called n-th Cremona group.

Thm we proved $\Rightarrow$ $\mathrm{Bir}\, \mathbb{P}^n \simeq \mathrm{Gal}_k\, k(x_1, \dots, x_n)$.

By Remark, $PGL_{n+1}(k) \leq \mathrm{Bir}\, \mathbb{P}^n$

$\quad = GL_{n+1}(k)/k^\times$ – acts on $\mathbb{P}^n$ via linear transformations

Ex. 1) $\mathrm{Bir}\, \mathbb{P}^1 \simeq PGL_2(k)$ (prove later)

2) $\mathrm{Bir}\, \mathbb{P}^2$ is generated by $PGL_3(k)$ and the Cremona transformation, so only linear + quadratic transforms — Max Noether's thm

3) $\mathrm{Bir}\, \mathbb{P}^n$ is huge when $n \geq 3$ (no simple description, uncountably many generators)

**Prop.** Rational maps $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ can always be extended to morphisms.

**Cor.** $\mathrm{Bir}\,\mathbb{P}^1 = \mathrm{Aut}\,\mathbb{P}^1 = PGL_2(k)$

prove later in the course

**Proof:** $\varphi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$, $\quad U_\varphi$ – maximal subset of def.
Let $V := \varphi^{-1}(\varphi(U_\varphi) \cap D_+(x_0)) \subseteq \mathbb{P}^1$ open,
we get a morphism

$$\varphi|_V : V \to D_+(x_0) \simeq \mathbb{A}^1,$$

which in turn corresponds to

$$k[t] \to \mathcal{O}_V(V) \subset k(\mathbb{P}^1)$$

$$t \mapsto \frac{f_1}{f_0},$$

where $f_0, f_1 \in k[y_0, y_1]$ homog. of deg $d$ without common factors, and $f_0 \neq 0$ on $V$

Define $\mathbb{P}^1 \xrightarrow{\Phi} \mathbb{P}^1$ as $(y_0 : y_1) \mapsto (f_0(y_0, y_1) : f_1(y_0, y_1))$.

- $\Phi$ is a morphism because $f_0, f_1$ have no common factors
- $\Phi$ extends $\varphi$: $x_0 \neq 0 \Rightarrow \varphi = \frac{f_1}{f_0}$

**Rem.** By the same argument, any $\varphi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ extends to $\Phi : \mathbb{P}^1 \to \mathbb{P}^n$.

Ex. $Q = Z_+(xw - zy) \subset \mathbb{P}^3$ quadric surface

- $Q \underset{bir}{\sim} \mathbb{P}^2$ : $\varphi : Q \underset{\leftarrow - -}{- - \rightarrow} \mathbb{P}^2 : \psi$

$$(x:y:z:w) \mapsto (x:y:z)$$
$$\left(x:y:z:\frac{yz}{x}\right) \leftarrow\!\shortmid (x:y:z)$$

- $Q \not\cong \mathbb{P}^2$: on $\mathbb{P}^2$ any 2 lines intersect, but on $Q$ we have disjoint lines $Z_+(x,y)$ and $Z_+(z,w)$

$\Rightarrow Q$ and $\mathbb{P}^2$ are not homeomorphic.

- $\mathrm{Bir}\, Q \cong \mathrm{Bir}\, \mathbb{P}^2$

$$\begin{array}{ccc} \mathrm{Aut}\, Q & \not\cong & \mathrm{Aut}\, \mathbb{P}^2 \\ \shortparallel & & \shortparallel \\ \mathrm{Aut}\, \mathbb{P}^1 \times \mathbb{P}^1 & & PGL_3(k) \\ \shortparallel & & \\ \mathbb{Z}_2 \ltimes PGL_2(k) \times PGL_2(k) & & \end{array}$$

# § Birational classification

Giant goal: classify varieties up to isom

$\cup$

Big goal: $-\|-$ up to birationality

$\cup$

Rationality problem: detect which varieties are rational, i.e. birational to $\mathbb{P}^n$.

Hard problem even for $Z(f) \subset \mathbb{P}^{n+1}$ regular hypersurface.

To work on this problem, one constructs birational invariants and tries to compute them.

such as $\operatorname{Bir} X$

Ex. 1) most varieties we've seen were rational, since they had a parametrization.

E.g. $\quad Gr(2,n) \underset{bir}{\sim} \mathbb{P}^{2n-4}$

2) simplest example of a non-rational variety is Fermat cubic

$Z(x^3+y^3+z^3) \subset \mathbb{P}^2$ (maybe prove later)

3) $X = Z(f) \subset \mathbb{P}^4$ degree 4 hypersurface

Iskovskikh-Manin: 1970 $\quad \operatorname{Bir} X$ is finite $\Rightarrow$ X not rational

## § Blow-ups

<u>NB</u>: varieties are no longer assumed irreducible

Blow-ups: important examples of birational morphisms! (e.g., between projective varieties)

„Blow up = zoom in, i.e. replace a subvariety with its tangent directions"

<u>Idea</u>: to „blow up" a closed subvariety without changing its open complement, i.e. for $Z \subset X$ closed construct a surjective birational map $\pi : \tilde{X} \to X$ s.t. $\pi^{-1}(X - Z) \longrightarrow X - Z$ is an isom.

For example, one could choose $Z \subseteq \mathrm{Sing}(X)$ in such a way that $\tilde{X}$ is „less singular", and in char 0 one can always obtain a regular variety by a finite sequence of blow-ups, this procedure is called „resolution of singularities" of $X$:

$$\underset{\text{regular}}{\tilde{X}_N} \to \dots \to \tilde{X}_2 \to \tilde{X}_1 \to \underset{\text{singular}}{X}$$

Blow-ups

it's a very hard thm that resolution exists in char 0!

Ex: blow-up along a point in $\mathbb{P}^2$

Recall: $\bullet \forall p \neq q \in \mathbb{P}^2$ there is a unique line $\ell \subset \mathbb{P}^2$ connecting them

$\bullet$ lines in $\mathbb{P}^2$ that pass through a fixed point $p$ are parametrized by $\mathbb{P}^1$. Say, $p=(0:0:1)$, then the „plane at $\infty$" is $H=\{(x:y:0)\} \simeq \mathbb{P}^1$ – plane that doesn't contain $p$, and the bijection sends $\ell \ni p$ to $\ell \cap H \in \mathbb{P}^1$.

We can assume $p=(0:0:1) \in \mathbb{P}^2$.

Let $X = \mathbb{P}^2$, $\tilde{X} := Z_+(x_0 y_1 - x_1 y_0) \subset \underset{x_0:x_1:x_2}{\mathbb{P}^2} \times \underset{y_0:y_1}{\mathbb{P}^1}$

Exercise. $\tilde{X} = \{(q, \ell) \mid q \in X, \ell \in \underset{\text{lines through } p}{\mathbb{P}^1}, \ell \subset \mathbb{P}^2 \text{ contains } p \text{ and } q\}$

$\Downarrow$ intuitively clear

Prop. Let $\pi: \tilde{X} \to X = \mathbb{P}^2$ – projection.

① $E := \pi^{-1}(p) \cong \mathbb{P}^1$

② $\pi|_{\tilde{X}-E} : \tilde{X} - E \to X - p$ is an isom,

so $\pi$ is birational surjective.

Proof: ① clear by construction

② consider the rational map

$$\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \qquad (x_0:x_1:x_2) \mapsto (x_0:x_1)$$

We define the inverse to $\pi|_{\tilde{X}-E}$ as

$$X-p \longrightarrow \mathbb{P}^2\times\mathbb{P}^1 \qquad x \mapsto (x, \varphi(x))$$

The image lies in $\tilde{X}$ because in coordinates

$(x_0:x_1:x_2) \mapsto (x_0:x_1:x_2)\times(x_0:x_1)$ – satisfies the equation.

Moreover $\forall q \in \tilde{X}$ is in the image because

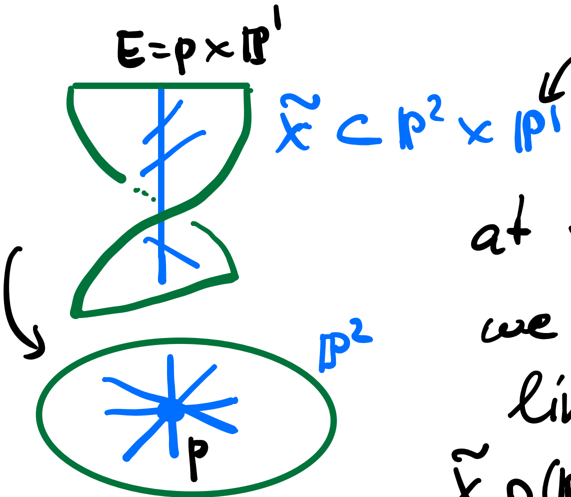
$$q = (x_0:x_1:x_2)\times(y_0:y_1) \in \underbrace{D_+(x_0)}_{\text{can assume } D_+(x_0) \text{ or } D_+(x_1) \text{ on } X-p}\times\mathbb{P}^1$$

$$(y_0:y_1) = (x_0y_0 : x_0y_1) = (x_0y_0 : x_1y_0) = (x_0:x_1)$$

$x_0 \neq 0 \Rightarrow y_0 \neq 0$

Illustration:

„staircase"




lines through p

at the level $\{\ell\}\in\mathbb{P}^1$ we have the line $\ell\subset\mathbb{P}^2$:

$$\tilde{X}\cap(\mathbb{P}^2\times\{\ell\}) = \ell\times\{\ell\}$$

„E is the projective cone of tangent directions at p":

E is parametrized by all directions in which p can move.

# General blow-ups

$X \subseteq \mathbb{A}^n$ affine variety, $Z \subset X$ closed, $U = X - Z$,
$I(Z) = (f_1, \ldots, f_r) \subset k[X]$.

Define $f : U \longrightarrow \mathbb{P}^{r-1}$
$x \longmapsto (f_1(x) : \ldots : f_r(x))$

Consider the graph
$\Gamma_f := \{(x, f(x))\} \subset U \times \mathbb{P}^{r-1} \subset X \times \mathbb{P}^{r-1}$

def. The blow-up $Bl_Z X$ of $X$ along $Z$ is

$$\tilde{X} := \overline{\Gamma_f} \subset X \times \mathbb{P}^{r-1} \quad \text{(closure in } X \times \mathbb{P}^{r-1})$$

$$\pi \downarrow \text{ - projection}$$

$$X$$

there's a different construction in the language of schemes, with a universal property!

The exceptional locus is
$E := \pi^{-1}(Z)$.

Rem. $\Gamma_f \subset U \times \mathbb{P}^{r-1}$ is closed because $\mathbb{P}^{r-1}$ is separated, hence

$\pi : \tilde{X} - E \xrightarrow{\sim} X - Z$ is an isom,

so $\pi$ is birational.

**def.** Let $Y \subset X$ closed subvariety s.t. $Y \not\subseteq Z$.
The strict transform of $Y$ is
$$\tilde{Y} := Bl_{Y \cap Z} Y = \overline{\Gamma_{f|_{Y-Z}}} = \overline{\pi^{-1}(Y-Z)} \subset \tilde{X}$$
The total transform of $Y$ is $\pi^{-1}(Y)$:
we have
$$\pi^{-1}(Y) = \tilde{Y} \cup e$$
– extra piece

where $e = \pi^{-1}(Y) \cap \underset{\pi^{-1}(Z)}{\overset{\shortparallel}{E}}$ satisfies $\pi(e) \subseteq Y \cap Z \subsetneq Y$.
„e is not too big"

Now, let $(x,y) \in \Gamma_f \subset U \times \mathbb{P}^{r-1}$.
By construction, $(x,y) = (x_1, \dots, x_n) \times (y_1 : \dots : y_r)$
where $(y_1, \dots, y_r)$ is proportional to $(f_1(x), \dots, f_r(x))$.

That means, the matrix
$$M = \begin{pmatrix} f_1(x) & \dots & f_r(x) \\ y_1 & & y_r \end{pmatrix}$$
has rank 1 on $\Gamma_f$, because
$$f_i(x) \cdot y_j - f_j(x) \cdot y_i = 0 \quad i \neq j.$$
These equations also hold on $\overline{\Gamma_f}$,
so we get:

**Prop.** The blow-up $\tilde{X} \subset X \times \mathbb{P}^{r-1}$ satisfies
$$\tilde{X} \subseteq \left\{ (x,y) \in X \times \mathbb{P}^{r-1} \,\middle|\, \operatorname{rk} \begin{pmatrix} f_1(x) & \dots & f_r(x) \\ y_1 & \dots & y_r \end{pmatrix} \leq 1 \right\}$$
[actually =

notion from CA

⟺ when $k[X]$ is Cohen-Macaulay

**Fact:** If $I(Z)=(f_1,\dots,f_r)$ s.t. $(f_1,\dots,f_r)$ is a "regular sequence" ($\Rightarrow$ codim $Z=r$), then $\tilde{X}$ is fully determined by these equations, i.e. $\subseteq$ in the Prop. is $=$.

**Ex:** Blow-up of $\mathbb{A}^n$ at the origin

We have $0\in\mathbb{A}^n \hookleftarrow U=\mathbb{A}^n-0$

and $f: U \longrightarrow \mathbb{P}^{n-1}$

$$x \longmapsto (x_1:\dots:x_{n-1})$$

The graph $\Gamma\subset\mathbb{A}^n\times\mathbb{P}^{n-1}$ consists of $(x_1,\dots,x_n)\times(y_1:\dots:y_n)$ s.t.

$$M=\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n\end{pmatrix} \text{ has rank } 1.$$

Let $W\subset\mathbb{A}^n\times\mathbb{P}^{n-1}$ be the subvariety defined by vanishing minors of $M$.

We know that $Bl_0\mathbb{A}^n\subseteq W$.

**Claim:** $Bl_0\mathbb{A}^n = W$.

Enough to show:

$W$ is irreducible and $\dim W=n$, because so is the blow-up:

$Bl_0\mathbb{A}^n=\overline{\Gamma_f}$ and $\Gamma_f\cong\mathbb{A}^n-0 \Rightarrow Bl_0\mathbb{A}^n$ is irred of dimension $n$.

To do so, we will show that $W$ has open covering by $U_i \simeq \mathbb{A}^n$ s.t. $U_i \cap U_j \neq \emptyset$.

**Lemma.** $W = \bigcup_i U_i$ open covering of a top. space, $U_i$ irreducible $\forall i$, $U_i \cap U_j \neq \emptyset$ $\forall i,j$. Then $W$ is irreducible.

**Proof:** exercise :)

Let $pr: \mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ and $V_1 := pr^{-1}(D_+(y_1))$.

We have:

$$V_1 \simeq \mathbb{A}^n \times \mathbb{A}^{n-1} \simeq \mathbb{A}^{2n-1}$$

can assume $y_1 = 1$

Then $W \cap V_1$ is defined by equations

$x_i = y_i x_1 \quad i = 2, \dots, n,$ because

$$x_i y_j = y_i x_1 y_j = y_i x_j \quad \forall i \neq j.$$

Hence $W \cap V_1 \simeq \mathbb{A}^n$, and same for other charts $V_i$.

The sets $U_i = \{W \cap V_i\}_{i=1}^n$ form an open covering of $W$ by $\mathbb{A}^n$'s, and $U_i \cap U_j \neq \emptyset$ $\forall i,j$. ∎

Ex: blow-up of a cuspidal cubic

$C = Z(y^2 - x^3) \subset \mathbb{A}^2$

$X := \mathbb{A}^2$, $\tilde{X} := Bl_p \mathbb{A}^2$, $p = (0,0)$.

$\tilde{X} = \{ (x, y ; u_0 : u_1) \mid u_1 x - u_0 y = 0 \} \subset \mathbb{A}^2 \times \mathbb{P}^1$

The total transform of $C$ is

$$\pi^{-1}(C) = \left\{ \begin{array}{l} u_1 x - u_0 y = 0 \\ y^2 - x^3 = 0 \end{array} \right\} \subset \mathbb{A}^2 \times \mathbb{P}^1$$

As we saw, $\pi^{-1}(C) = \underbrace{\tilde{C}}_{\text{strict transform}} \cup \underbrace{E}_{\text{exceptional locus}}^{\neq \emptyset \text{ because } C \ni p}$

Consider the chart $U_0 = \{u_0 = 1\}$:

$$\left\{ \begin{array}{l} u_1 x - y = 0 \\ y^2 - x^3 = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} y = u_1 x \\ u_1^2 x^2 - x^3 = \underbrace{x^2}_{E} \underbrace{(u_1^2 - x)}_{\tilde{C}} = 0 \end{array} \right.$$

Hence $\tilde{C} \cap U_0$ is the conic

$Z(u_1^2 - x) \subset \mathbb{A}^2$ – regular variety.

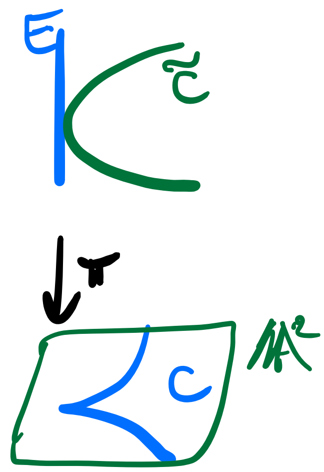
In the chart $U_1 = \{u_1 = 1\}$:

$$\left\{ \begin{array}{l} x = u_0 y \\ y^2 - u_0^3 \cdot y^3 = \underbrace{y^2}_{E} \underbrace{(1 - u_0^3 y)}_{\tilde{C}} = 0 \end{array} \right.$$

Hence $\tilde{C} \cap U_1 = Z(1 - u_0^3 y) \subset \mathbb{A}^2 \Rightarrow \tilde{C}$ is regular!

# Pictures

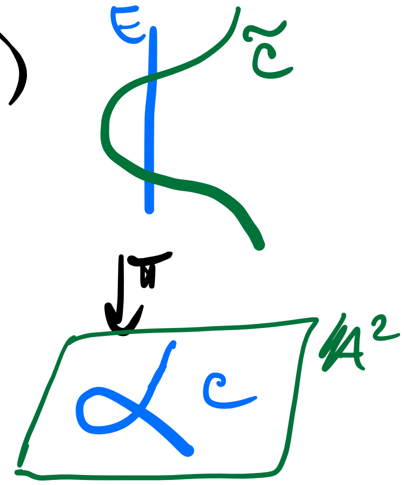
I)




$C = \{y^2 = x^3\}$

$C$ is singular,
$\pi|_{\tilde{C}}$ is bijective
but not isom

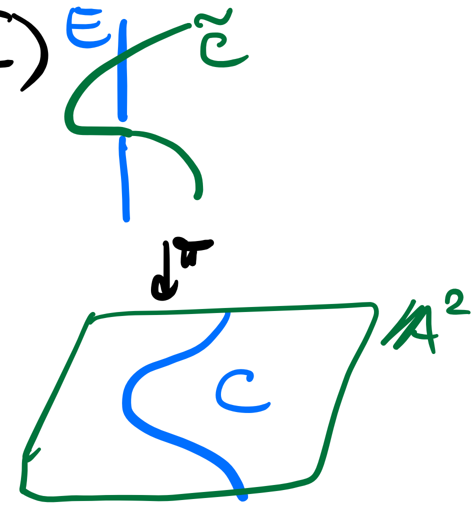
II)




$C = \{y^2 = x^3 + x^2\}$

$C$ is singular,
$\pi^{-1}(0)$ is 2 points,
$\pi|_{\tilde{C}}$ isom outside 0

III)




$C = \{y^2 = x^3 + x\}$

$C$ is regular,
$\pi|_{\tilde{C}}$ is isom.
(no coincidence!)

NB: Blow-up of a singularity can be singular!

Ex. $C = Z(y^2 - x^5) \subset \mathbb{A}^2$

$$\tilde{X} = Bl_0 \mathbb{A}^2 = Z(u_1 x - u_0 y) \subset \mathbb{A}^2 \times \mathbb{P}^1$$

In the chart $U_0 = \{u_0 = 1\}$:

$$\begin{cases} y = u_1 x \\ y^2 = x^5 \end{cases} \Rightarrow \pi^{-1}(C) \cap U_0 \simeq Z(\underbrace{x^2}_{E}\underbrace{(u_1^2 - x^3)}_{\tilde{C}}) \subset \mathbb{A}^2$$

$\Rightarrow \tilde{C}$ is the cuspidal cubic!