Chapter 3
Properties of morphisms

Morphisms are the main tool for studying algebraic varieties: you try to map your variety into something simpler, with fibers that we can understand.

$\varphi : X \to Y \rightsquigarrow$ properties of fibers $\varphi^{-1}(y)$? image $\varphi(X)$?

General principle: a morphism is not too bad on a dense open set, complications happen on a proper closed subset.

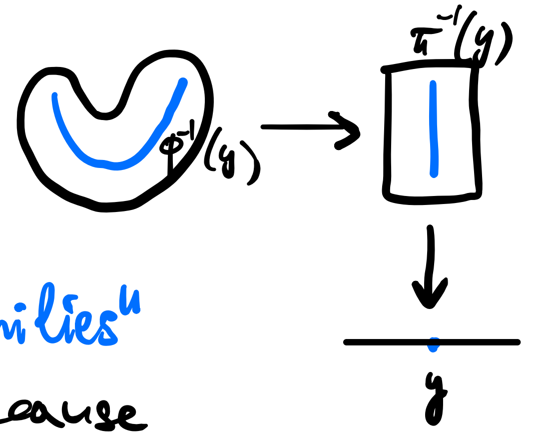
§ Fibers

Generic structure thm

$\varphi : X \to Y$ dominant map, X, Y affine irreducible $\Rightarrow$
$\exists$ open affine subset $U \subseteq Y$ s.t.
$\varphi : \varphi^{-1}U \to U$ factors as

$$\varphi^{-1}U \xrightarrow{\psi} \mathbb{A}^n \times U \xrightarrow{\pi} U$$

finite surjective — projection



„Noether normalization in families"

Rem. $n = \dim X - \dim Y$, because ψ finite surjective.

Proof. Let $K := k(Y)$.

Let $A := k[Y]$, $B := k[X]$, φ dominant $\Rightarrow$
$B \supseteq A$ and B is a f.g. A-algebra:
B is generated by $b_1, \dots, b_s$ as A-algebra
$\Rightarrow B_K := B \otimes_A K$ is a f.g. K-algebra.

By Noether normalization lemma,
$\exists$ alg. indep. $\omega_1, \dots, \omega_n \in B_K$ s.t.
$B_K \supseteq K[\omega_1, \dots, \omega_n]$ is a fin. gen. module.
Hence $b_1 \otimes 1, \dots, b_s \otimes 1$ are integral over $K[\omega_1, \dots, \omega_n]$
as elements of a finite module.

Let h be the product of all denominators
in ω_i's and coeffs of monic equations
of $b_j \otimes 1$'s (coeffs belong to $K[\omega_1, \dots, \omega_n]$),
Then $h \in A$. We have:

$D(h) \subset Y$ is affine, $k[D(h)] \cong A_h$;
$\varphi^{-1}(D(h)) = D(\varphi^* h)$, $k[\varphi^{-1} D(h)] \cong B_h$.

We define $U := D(h)$.

We have:

- $b_i \in B_h$ and they generate B_h
as an A_h-algebra,

and they are integral over $A_h[\omega_1,..,\omega_n]$

$\Rightarrow B_h$ is a finite $A_h[\omega_1,..,\omega_n]$ - algebra

our choice of h

. ω_i's are alg independent over A_h.

We get:

$$\begin{tikzcd} B_h & A_h[\omega_1,..,\omega_n] \arrow[l, hook', "\text{fin.alg.}"'] \\ & A_h \arrow[u, hook] \arrow[ul] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \varphi^{-1}\mathcal{U} \arrow[r] \arrow[dr] & \mathbb{A}^n \times \mathcal{U} \arrow[d] \\ & \mathcal{U} \end{tikzcd}$$

Cor. $\varphi: X \to Y$ dominant, X, Y irreducible $\Rightarrow$ $\exists$ affine open $\mathcal{U} \subseteq Y$ and a covering $\varphi^{-1}(\mathcal{U}) = \bigcup_{i=1}^{m} V_i$ by affine opens s.t.

$\varphi|_{V_i} = \pi \circ \psi_i$ where $\psi_i : V_i \to \mathbb{A}^n \times \mathcal{U}$ is finite surjective.

Proof: can assume Y affine (replace Y with affine and X with preimage)

$X = \bigcup_{i=1}^{m} W_i$ any affine open cover

$\hookrightarrow \exists \mathcal{U}_i \subseteq Y$ s.t. $\varphi|_{\varphi^{-1}(\mathcal{U}_i)} : \varphi^{-1}(\mathcal{U}_i) \xrightarrow{\psi_i} \mathbb{A}^n \times \mathcal{U}_i \xrightarrow{\pi} \mathcal{U}_i$

$\hookrightarrow \mathcal{U}$: affine open inside $\cap\, \mathcal{U}_i$

$\hookrightarrow \varphi^{-1}\mathcal{U} = \bigcup_{i=1}^{m} V_i$ where $V_i := \varphi^{-1}\mathcal{U} \cap W_i$.

The open sets V_i are affine, because $\varphi|_{V_i} = \pi \circ \psi_i$ is an affine morphism $\forall i$.

Cor. $\varphi: X \to Y$ dominant morphism $\Rightarrow$
$\exists\, U \subseteq \varphi(X)$ s.t. $U \subseteq Y$ is open.

Proof: take $V \xrightarrow{\psi} \mathbb{A}^n \times U \xrightarrow{\pi} U$ as in the thm
$\Rightarrow \pi(\psi(V)) = U$ because ψ is
dominant + finite $\Rightarrow$ surjective, and π is surj.

Ex. $\mathbb{A}^2 \longrightarrow \mathbb{A}^2$
$(x, y) \overset{\varphi}{\longmapsto} (x, xy)$ $\Rightarrow \varphi(\mathbb{A}^2) = D(x) \cup 0$
$\cap$ $\mathbb{A}^2$

neither open nor closed

One can take $D(x) \underset{\varphi}{\xrightarrow{\sim}} D(x) \subset \mathbb{A}^2$
(here $n = 0$) open

In fact, when $n = 0$ a stronger statement is true.

Fact $\varphi: X \to Y$ dominant morphism, X, Y irreducible, $\dim X = \dim Y$.
Then $\exists$ dense open $U \subseteq Y$ s.t.
$\varphi: \varphi^{-1}U \to U$ is finite.

(it's not obvious because finiteness is <u>not</u> open on the source, so you can't glue finite maps)

Recall: $\varphi : X \to Y$ dominant morphism, X, Y irreducible, $y \in \varphi(X)$ $\Rightarrow$

$$\dim Z \geq \dim X - \dim Y$$

$\forall$ component $Z \subseteq \varphi^{-1}(y)$.

Ex. $\pi : Bl_p \mathbb{P}^2 \to \mathbb{P}^2$

$\dim \mathbb{P}^2 = \dim Bl_p \mathbb{P}^2 = 2$

$\dim \pi^{-1}(p) = \dim \mathbb{P}^1 = 1 > 0$

strict inequality

Idea: the equality holds „generically"

Thm. $\varphi : X \to Y$ dominant, X, Y irreducible $\Rightarrow \exists$ $U \subseteq Y$ open set s.t. $\forall y \in U$

$$\dim Z = \dim X - \dim Y$$

relative dimension of φ

$\forall$ component $Z \subseteq \varphi^{-1}(y)$.

Proof. Let $U \subseteq Y$ as in the generic structure thm. (Cor.)

So $\varphi^{-1}(U) = \bigcup V_i$ s.t. $V_i \to U$ factors as

$$V_i \xrightarrow{\text{finite}} \mathbb{A}^n \times U \xrightarrow{\text{projection}} U$$

Since $y \in U$, $\exists\, i : V_i \cap Z \neq \emptyset$ $\Rightarrow$

$$\dim Z = \dim Z \cap V_i = \dim (\varphi|_{V_i})^{-1}(y).$$

We have $\varphi|_{V_i \cap Z} : \underset{\substack{\text{finite} \\ \text{dominant}}}{V_i \cap Z \longrightarrow} \mathbb{A}^n \times \{y\} \longrightarrow y$

$\Rightarrow \dim V_i \cap Z = \dim \mathbb{A}^n \times \{y\} = n = \dim X - \dim Y.$

Intuition: in general, fibers can be complicated and have components of different dimensions, but as $y \in Y$ varies, $\dim \varphi^{-1}(y)$ can only „jump up",
i.e. the function
$y \mapsto \dim \varphi^{-1}(y)$ is upper semicontinuous, which means:

Prop. Let $\varphi : X \to Y$ morphism of irred. varieties.
Let $\omega_r(\varphi) := \{ y \in Y \mid \dim \varphi^{-1}(y) \geq r \}$
be a stratification of Y: $\omega_n \subseteq \omega_{n-1} \subseteq \ldots \subseteq \omega_{-1} = Y.$

Then $\omega_r(\varphi) \subseteq Y$ is closed $\forall r$.

Proof. Replace Y with $\overline{\varphi(X)}$ $\Rightarrow \varphi$ dominant.

If $r \leq \dim X - \dim Y$, $\omega_r(\varphi) = Y$.
In general, induction on $\dim Y$.
If $\dim Y = 0$, $\omega_r(\varphi) = Y$ or $\emptyset$.
Assume $\dim Y = d > 0$ and $r > \dim X - \dim Y$.

Pick $\mathcal{U} \subseteq Y$ open s.t. $\forall y \in \mathcal{U}$ $\forall$comp. $T \subseteq \varphi^{-1}(y)$
$\dim T = \dim X - \dim Y$ (from Thm above)

Then $Y - \mathcal{U} = \bigcup_{i=1}^{n} Z_i$, Z_i closed irred, $\dim Z_i < d$.
$\forall i$ $\varphi^{-1}(Z_i) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} Z_{ij}$, $Z_{ij} \subseteq X$ closed irred.

We get $\omega_r(\varphi) = \bigcup_{i,j} \omega_r(\varphi|_{Z_{ij}})$,
and each $\omega_r(\varphi|_{Z_{ij}}) \subseteq Z_i$ is closed by induction, hence
$\omega_r(\varphi)$ is a finite union of closed subsets.

§ Images

def. X top. space. A subset $Y \subseteq X$ is **locally closed** if $Y \subseteq U$ is closed where $U \subseteq X$ is open.

For varieties: if $Y = W \cap U$ where $W \subseteq X$ is closed and $U \subseteq X$ is open.

A **constructible subset** is a finite union of locally closed subsets.

Ex. $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$
$(x,y) \mapsto (x, xy)$

$$Y := \varphi(X) = (\mathbb{A}^2 - Z(x)) \cup 0 = D(x) \cup Z(y)$$

is a constructible set.

Chevalley's thm.
$\varphi : X \to Y$ morphism of varieties
$\Rightarrow \varphi(X) \subseteq Y$ is constructible.

moral: the image of a polynomial map $\mathbb{A}^n \to \mathbb{A}^m$ can be described via polynomial equations $g_1 = \dots = g_r = 0$ and polynomial inequations $h_1 \neq 0, \dots, h_s \neq 0$.

Proof. Enough to prove for X, Y irreducible, because we can check on every component of X, and image of irreducible is irreducible.

Replace Y by $\overline{\varphi(X)}$ => assume φ is dominant.
Induction on $\dim Y$.
We know from the Cor. of the Generic structure thm:
$\exists U \subseteq Y$ open: $U \subseteq \varphi(X)$.

Then $Y \setminus U = \bigcup_{i=1}^{n} Z_i$ Z_i closed irred, $\dim Z_i < \dim Y$, and $\varphi^{-1}(Z_i) = \bigcup_{i=1}^{?} \bigcup_{j=1}^{m} Z_{ij}$; $Z_{ij} \subseteq Z_i$ closed irred.

By induction, $\varphi|_{Z_{ij}} : Z_{ij} \to Z_i$ has a constructible image =>
$$\varphi(X) = \underbrace{U}_{\text{constructible}} \cup \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \underbrace{\varphi(Z_{ij})}_{\text{constructible}}$$

„Images of projective varieties are closed"

Thm. $\varphi : X \to Y$ morphism of varieties.
If $Z \subseteq X$ is projective, then $\varphi(Z) \subseteq Y$ is closed.
We skip the proof.

„Projective spaces are universally closed"

Cor. X alg variety, $Z \subseteq X \times \mathbb{P}^n$ closed subset, $\pi : X \times \mathbb{P}^n \to X$ projection.
Then $\pi(Z)$ is closed.

Proof: reduce to the case X projective

$\pi(Z)$ closed $\Leftrightarrow \pi(Z) \cap U_i \subseteq U_i$ closed for $X = \bigcup U_i$ open cover

$\Rightarrow$ can assume that X is open in a projective variety W $(X \subseteq \mathbb{A}^m \Rightarrow W := \overline{X} \subseteq \mathbb{P}^m)$

Denote $\overline{Z} :=$ closure of Z in $W \times \mathbb{P}^n$.

By Thm, $\pi(\overline{Z}) \subseteq W$ is closed.

Claim: $\pi(Z) = X \cap \pi(\overline{Z})$ (hence $\pi(Z) \subseteq X$ closed)

$\subseteq$ ok

$\supseteq$: $Z = \overline{Z} \cap (X \times \mathbb{P}^n)$ since $Z \subseteq X \times \mathbb{P}^n$ closed

$\Rightarrow \forall x \in X$

$(\{x\} \times \mathbb{P}^n) \cap \overline{Z} = \{x\} \times \mathbb{P}^n \cap \overline{Z} \cap X \times \mathbb{P}^n = \{x \times \mathbb{P}^n\} \cap Z$,

so $X \cap \pi(\overline{Z}) \subset \pi(Z)$.

Rem. For a topological space, universally closed $\Leftrightarrow$ (quasi)compact, so $\mathbb{P}^n$ and other projective varieties in Var_k are analogues of compact spaces in Top.

§ Disclaimer

„Underlying top. space is a red herring!"

$\Rightarrow \mathcal{O}_X$ is important!

The forgetful functor
$$F: \mathrm{Var}_k \to \mathrm{Top}$$
changes many properties of objects!

So, $F(X)$ does not know much about X.
Here are some examples.

① F does not preserve products:
$$F(\underbrace{\mathbb{A}^1 \times \mathbb{A}^1}_{\overset{\shortparallel}{\mathbb{A}^2}}) \neq F(\mathbb{A}^1) \times F(\mathbb{A}^1)$$

② F does not distinguish different curves:
$$F(C) \simeq F(C') \quad \forall \text{ curves } C, C'$$
(any bijection respects cofinite topology)

③ almost all varieties X behave like Hausdorff spaces (they are separated), but their underlying spaces $F(X)$ are not Hausdorff

④ all varieties X have quasi-compact underlying spaces $F(X)$, but only projective X behave like compact spaces: are universally closed

§ Projective morphisms

def. A morphism $\varphi: X \to Y$ is called **projective** if there is a factorization

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "\varphi"'] & Y \times \mathbb{P}^n \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

π – projection

where i is a closed embedding (exhibits X as a closed subvariety)

„X is fibered in projective varieties over Y"

Rem. Cor. $\Rightarrow$ projective morphisms are closed, i.e. $\varphi(\text{closed}) = \text{closed}$.

Thm. Let Y be affine variety, $A := k[Y]$, and $\varphi: X \to Y$ a projective morphism. Then φ^* makes $\mathcal{O}_X(X)$ a f.g. A-module.

ring of global regular functions on X

Rem. $Y = pt \rightsquigarrow \varphi$ projective iff X projective $\Rightarrow$ can deduce that $\mathcal{O}_X(X) \cong k$ for connected X.

Proof:

Fact from CA: enough to show that $\mathcal{O}_X(X)$ is an integral extension of A, because $\mathcal{O}_X(X)$ is a f.g. A-algebra.

We pick $f \in \mathcal{O}_X(X)_g$ want to find a monic equation for f with coeffs in A.

Let $U := D(f) \Rightarrow f^{-1} \in \mathcal{O}_X(U) \rightsquigarrow f^{-1} : U \to \mathbb{A}^1$.

Then $\Gamma := \{(x, f^{-1}(x)\} \subset U \times \mathbb{A}^1$ is closed.

<u>Claim</u>: $\Gamma \subset X \times \mathbb{A}^1$ is also closed.

This is because Γ is the preimage of

$H = Z(ut-1) \subset \mathbb{A}^1_u \times \mathbb{A}^1_t$ – hyperbola (closed subset)

under $f \times id : X \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \times \mathbb{A}^1$:

$(f(x), t) \in H$ iff $t = f(x)^{-1}$.

Now consider $\theta := \varphi \times id : X \times \mathbb{A}^1 \to Y \times \mathbb{A}^1$ – projective morphism

By Cor., $\theta(\Gamma) \subset Y \times \mathbb{A}^1$ is closed $\Rightarrow$

$\theta(\Gamma) = Z(\mathfrak{a})$ for $\mathfrak{a} \subset A[t]$ an ideal.

We have: $\theta(\Gamma) \cap (Y \times \{0\}) = \emptyset$, because $f^{-1}(x) \neq 0$.

By Nullstellensatz, $\exists\, F \in \mathfrak{a}$, $G \in A[t]$:

$(*)\ F(t) + t \cdot G(t) = 1$.

Consider the image of $(*)$ under $A \xrightarrow{\varphi_*} \mathcal{O}_X(U)$.

We get: $F(f^{-1}) = 0$ because $\theta(\Gamma) = Z(\mathfrak{a})$.

We obtain: $1 = G(f^{-1}) \cdot f^{-1}$ in $\mathcal{O}_X(U)$.

Multiplying with f^N, we get an integral dependence for f over A in $\mathcal{O}_X(U)$.

Multiply by f again $\rightsquigarrow$ get monic equation in $\mathcal{O}_X(X)$.

def. A morphism is quasifinite if all the fibers are finite.

Rem. finite $\Rightarrow$ quasifinite
hard to check easy to check

Thm. A projective quasifinite morphism is finite.

Rem. In fact, finite = projective quasifinite for varieties.

Proof: Let $\varphi : X \overset{i}{\hookrightarrow} Y \times \mathbb{P}^n \xrightarrow{\pi} Y$ be projective.

By previous Thm, left to show:
$\forall y \in Y$ has an open affine U s.t. $\varphi^{-1}(U)$ is affine, because $\varphi|_{\varphi^{-1}(U)}$ is still projective.

We identify X and $i(X) \subseteq Y \times \mathbb{P}^n$.
Since $\varphi^{-1}(y)$ is finite, choose a hyperplane $H = Z_+(h) \subset \mathbb{P}^n$ s.t. $\underbrace{\varphi^{-1}(y)}_{\cap\ \{y\}\times\mathbb{P}^n \simeq \mathbb{P}^n} \cap H = \emptyset$.

Consider $X \cap (Y \times H) \subset Y \times \mathbb{P}^n$ — closed
$\underset{\text{Cor.}}{\Longrightarrow}$ $\pi(X \cap (Y \times H)) \subset Y$ is closed.

By the choice of H, $y \notin \pi(X \cap (y \times H)) \Rightarrow$
$\exists$ U affine open nbhd of y s.t.
$\pi(X \cap (y \times H)) \cap U = \emptyset$.

We get: $\varphi^{-1}(U) = X \cap (U \times \mathbb{P}^n) = X \cap (U \times D_+(h))$

$$\Rightarrow \varphi^{-1}(U) \underset{\text{closed}}{\subseteq} \underbrace{U \times D_+(h)}_{\text{affine}} \Rightarrow \varphi^{-1}(U) \text{ is affine.}$$

Zariski's Main Thm.

Every quasi-finite morphism $\varphi: X \to Y$ factors as

$$X \xhookrightarrow[j]{} W \xrightarrow[\psi]{} Y,$$

where j is an open embedding and ψ is finite.

Ex. Symmetric power map

$$\varphi: \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \to \mathbb{P}^d$$

$$(u_1:v_1) \times \ldots \times (u_d:v_d) \mapsto \text{list of coeffs of } (u_1+v_1t)\cdot\ldots\cdot(u_d+v_dt) \in k[t]$$

This map projective and qfinite:
$\#\varphi^{-1}(a) \leq d!$ (permute factors) $\Rightarrow$ φ is finite.

By the way, $\varphi: \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 / \Sigma_d \xrightarrow{\sim} \mathbb{P}^d$.

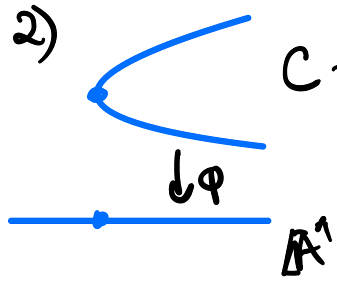
§ Degree

<u>def.</u> Let $\varphi: X \to Y$ dominant, X, Y irreducible, $\dim X = \dim Y$. The **degree** of φ is
$$\deg \varphi := [k(X) : k(Y)].$$

$\deg \varphi \in \mathbb{N}$ because $\operatorname{tr\,deg}_k k(X) = \operatorname{tr\,deg}_k k(Y)$, so $k(Y) \subseteq k(X)$ is a finite extension.

Lemma. $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ φ, ψ dominant, X, Y, Z irreducible, $\dim X = \dim Y = \dim Z$. Then $\psi \circ \varphi$ is dominant and
$$\deg(\psi \circ \varphi) = \deg \psi \cdot \deg \varphi .$$

Ex. 1) φ birational (e.g. blow-up)
$\varphi^*: k(Y) \xrightarrow{\sim} k(X) \Rightarrow \deg \varphi = 1$

2) 
$C = Z(x - y^2) \subset \mathbb{A}^2$

$k[C] = k[x,y]/_{x-y^2} \Rightarrow k(C) \simeq k(y)$

$k(\mathbb{A}^1) = k(x) \simeq k(y^2) \Rightarrow \deg \varphi = [k(y) : k(y^2)] = 2$

3) $\varphi: \mathbb{A}^1 \to \mathbb{A}^1 \quad x \mapsto f(x)$, $\quad \deg f = d$

$\Rightarrow \deg \varphi = [k(x) : k(f)] = d$

4) $\varphi: \mathbb{P}^1 \to \mathbb{P}^1 \quad x \mapsto (f(x) : g(x))$,

s.t. $f, g \in k[x]$ have no common zeroes.

Claim: $\deg \varphi = \max\{\deg f, \deg g\}$.

Proof: $\varphi^*: k(x) \to k(x)$

$$x \mapsto \frac{f(x)}{g(x)} =: h(x)$$

Then $\deg \varphi = [k(x) : k(h)]$.

Minimal poly for x over $k(h)$: $f(t) - g(t) \cdot h = 0$

irreducible (Gauss lemma)

Intuition: degree = number of pts in a generic fiber

fails in char p...

4) Frobenius morphism

char $k = p$ $\qquad \varphi: \mathbb{A}^1 \to \mathbb{A}^1$

$$x \mapsto x^p$$

$\Rightarrow \varphi^{-1}(a) = Z(x^p - a) = Z(x-a)^p$

char p

$\Rightarrow \# \varphi^{-1}(a) = 1 \ \forall a$

But $\deg \varphi = p$!

Reason: $k(x^p) \subset k(x)$ is not separable.

Thm (Generic freeness)
$\varphi: X \to Y$ finite dominant, X, Y irreducible

1) $\exists$ open affine $U \subseteq Y$ s.t. $V = \varphi^{-1}U$ is affine and $k[V]$ is a **free** $k[U]$-module of rank $= \deg \varphi$.

2) If $\varphi^*: k(Y) \subseteq k(X)$ is a separable extension, then $\exists$ open affine $U \subseteq Y$ s.t. $\forall y \in U$

always holds in char 0

$$\# \varphi^{-1}(y) = \deg \varphi.$$

Ex. $C = Z(x - y^2) \longrightarrow \mathbb{A}^1$
$(y, x) \longmapsto x$

$\forall \; x \neq 0 \quad \{x = y^2\}$ has 2 solutions $\Rightarrow \# \varphi^{-1}(x) = 2$

Let $U := D(x) \subset \mathbb{A}^1 \Rightarrow k[U] \simeq k[x]_x$

$$\underbrace{\varphi^{-1}(U) = D(\varphi^* x) = D(x)}_{=: V} \subset C$$

$$k[V] = \left({}^{k[x,y]}/_{(x - y^2)} \right)_x \cong \underbrace{k[x]_x \cdot 1 \oplus k[x]_x \cdot y}_{\text{free } k[x]_x\text{-module of rank } 2}$$