Serie 10

Exercise 1. Prove carefully the remaining assertions of Proposition 6.13.

Exercise 2. Let G be a locally compact abelian Hausdorff group. Show that $L^1(G)$ has always approximate units. Namely, use Lemma 6.15 to show that for every $f \in L^1(G)$ and $\varepsilon > 0$ there exists an open set $e \in V$ with the following property: if $u\colon G \to [0, \infty[$ is a Borel function which vanishes outside V and $\int_G u(x)d\mu(x) = 1$ then

$$\|f - f * u\|_1 < \varepsilon.$$

Let $|x|_p := e^{-v_p(x)}$ denote the p-adic norm on $\mathbb{Q}_p$ and recall that $\mathbb{Z}_p = \{ x \in \mathbb{Q}_p \mid |x|_p \leq 1 \}$. The ring of Adeles $\mathbb{A}_\mathbb{Q}$ of $\mathbb{Q}$ is defined as

$$\mathbb{A}_\mathbb{Q} := \left\{ (x_\infty, x_2, x_3, x_5, \ldots) \in \mathbb{R} \times \prod_{p \in \mathbb{P}} \mathbb{Q}_p \;\middle|\; |x|_p \leq 1 \text{ for all, except finitely many primes } p \right\}.$$

We endow $\mathbb{A}_\mathbb{Q}$ with a topology as follows: for every $S \subset \mathbb{P}$ finite consider

$$\mathbb{A}_S := \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p,$$

with the product topology, then $\mathbb{A}_\mathbb{Q} = \bigcup_{S \in \mathbb{P}} \mathbb{A}_S$ and we define $V \subset \mathbb{A}_\mathbb{Q}$ as open if and only if $V \cap \mathbb{A}_S$ is open in each $\mathbb{A}_S$ for every $S \subset \mathbb{P}$ finite.

Exercise 3. Show that $(\mathbb{A}_\mathbb{Q}, +)$ is a locally compact, abelian Hausdorff group. Moreover, show that there exists an injection

$$\begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{A}_\mathbb{Q}; \\ x & \longmapsto & (x, x, x, \ldots). \end{array}$$

Finally, show that $i(\mathbb{Q})$ is a discrete subgroup of $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_\mathbb{Q}/i(\mathbb{Q})$ is compact.