# Serie 3

**Exercise 1.** Verify that $C^n([0,1])$ with pointwise multiplication and norm

$$\|f\| := \sum_{k=0}^{n} \frac{\left\|f^{(k)}\right\|_\infty}{k!}$$

is a commutative unital Banach algebra for every integer $n$.

**Exercise 2.** Show that $C^\infty([0,1])$ does not admit any Banach algebra norm.

**Hint.** Suppose by contradiction that $\|.\|$ is a Banach algebra norm on $C^\infty([0,1])$.

1. Show that $C([0,1])$ with the infinity norm is a semisimple Banach algebra.
2. Use 1. to conclude that the inclusion $C^\infty([0,1]) \to C([0,1])$ is continuous so that there exists $c > 0$ with $\|f\|_\infty \leq c\,\|f\|$ for every $f \in C^\infty([0,1])$.
3. Use the Closed Graph Theorem and the previous steps to show that the derivation $D : C^\infty([0,1]) \to C^\infty([0,1])$ sending a function to its derivative is continuous.
4. Reach a contradiction studying the functions $t \mapsto \exp(\alpha t)$.

**Exercise 3.** Verify that the Guelfand topology is the weakest topology on $\hat{A}$ with respect to which all the functions $\hat{A} \to \mathbb{C}$ sending $\varphi$ to $\varphi(x)$ for $x$ in $A$ are continuous.

**Exercise 4.** Let $X$ be a locally compact Hausdorff space. Show that the bijection $X \to \widehat{C_0(X)}$ is a homeomorphism.

**Hint.** One way to solve the exercise is to use Urysohn's Lemma.